Threshold function for the edge connectedness of random bipartite graphs

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Random graphs constitute a big area in the modern graph theory; see, e.g., [1]. The Erdős–Rényi model (fixing the number $m(n)$ of the edges) and the Gilbert model (fixing the probability $\rho(n)$ of an edge) are standard models of the random graph. One of the main directions of the study of random graphs concerns threshold functions for different graph properties. Threshold functions are also considered for properties in particular classes of graphs, like bipartite graphs.

In [2], a combinatorial problem on words was reduced to finding the threshold function for the edge connectedness of random bipartite graphs in the Erdős–Rényi model. This function was found in [2] under some restrictions on the size p, q of the parts of the graph (we assume $p \geq q$). If one part is much smaller than the other (namely, $q = o(\frac{p}{\ln p})$), then $\phi(p,q) = \sqrt{pq(\ln q + O(1))}$ is the threshold function; if the sizes of the parts are close ($q = o(\frac{p}{\ln p})$), then the threshold function is $f(p,q) = \frac{pq}{p+q-2}\left(\ln \frac{pq}{p+q-2} + \ln\ln \frac{pq}{p+q-2} + O(1)\right)$.

The above results leave a range of growth rates of q uncovered. We analyze the behaviour of random bipartite graphs in this range. Our contribution is as follows.

1. The problem of searching the threshold function for the edge connectedness of random bipartite graphs in the Erdős–Rényi model is reduced to the same problem in the Gilbert model and vice versa. In particular, $\frac{f(p,q)}{pq}$ and $\frac{\phi(p,q)}{pq}$ are thresholds for the connectedness of random bipartite graphs in the Gilbert model in the same range of growth rates of q.

2. If $q = \alpha\frac{p}{\ln p} + o(\alpha\frac{p}{\ln p})$, $0 < \alpha < \ln\ln p$, then both $\phi(p,q)$ and $f(p,q)$ are strictly smaller than the threshold function; the same result applies for the Gilbert model.

3. If $\alpha < 1$, then $\phi(p,q) > f(p,q)$; replacing $O(1)$ in $\phi(p,q)$ with any $o(\ln q)$ function does not give the threshold function: the expected number of tree components with a single vertex in a smaller part and $(\frac{1}{\sqrt{\alpha}} - \alpha)\ln q$ vertices in a bigger part remains non-zero.

4. If $\alpha > 1$, then $f(p,q) > \phi(p,q)$; replacing $O(1)$ in $f(p,q)$ with any $O(\frac{(\ln\ln q)^2}{\ln\ln\ln\ln\ln q})$ function does not give the threshold function: the expected number of tree components with a single vertex in a smaller part and $2\ln\ln q$ vertices in a bigger part remains non-zero. Moreover, if $\alpha = O(1)$, replacing $O(1)$ in $f(p,q)$ with any $O(\frac{\ln q}{\ln\ln q})$ function does not give the threshold function: the expected number of tree components with a single vertex in a smaller part and $\frac{\ln p}{\ln\ln p}$ vertices in a bigger part remains non-zero.

5. If $\alpha < 1$, both symbolic and numerical computations support the following conjecture: the threshold function grows as $\beta\sqrt{pq\ln q}$, where $\beta > 1$ depends on α and is bounded.

References

[1] B. Bollobás, *Random Graphs*, second edition, Cambridge University Press, 2001.

[2] L. A. Idiatulina, A. M. Shur, Periodic Partial Words and Random Bipartite Graphs. *Fundamenta Informaticae.* **132(1)** (2014) 15–31.