Asymptotic approximation for the number of n-vertex graphs with given diameter

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Let $\mathcal{J}_{n,\,d=k},\ \mathcal{J}_{n,\,d\geq k},\ \mathcal{J}^*_{n,\,d\geq k}$ be the following classes of labeled n-vertex ordinary graphs: graphs of the diameter k, connected graphs of the diameter at least k, and graphs (not necessarily connected) with a shortest path of the length at least k, respectively. It is well known that almost all graphs have diameter 2. Consequently, the number of labeled n-vertex graphs of the diameter 2 is equal to the number $2^{\binom{n}{2}}$ of all n-vertex graphs asymptotically. This result was probably first established in [1]. For the number of graphs of a fixed diameter $k \geq 3$, the asymptotic formula $|\mathcal{J}_{n,\,d=k}| = 2^{\binom{n}{2}}(6\cdot 2^{-k}+o(1))^n$ as $n \to \infty$ was obtained in [2]. The same formula for the number $|\mathcal{J}_{n,\,d\geq k}|$ of connected graphs of the diameter at least k was established in [3]. However, this formula does not give an asymptotically exact value of the number of graphs in the classes $\mathcal{J}_{n,\,d=k},\ \mathcal{J}_{n,\,d\geq k}$ and error estimates in such asymptotic approximation. For any $k \geq 3$, the asymptotics of the number $|\mathcal{J}_{n,\,d=k}|$ of graphs with the fixed diameter k was found in [4] with an approximation error $r(n)$ satisfying the following estimates for all large enough n: $-c\left(\frac{9}{10}\right)^{n-k} \leq r(n) = O\left(k^2(n-k-1)^4\left(\frac{11}{12}\right)^{n-k-1}\right)$, where $c > 0$ is a constant independent of $n-k-1$.

It is obvious that $\mathcal{J}_{n,\,d=k} \subseteq \mathcal{J}_{n,\,d\geq k} \subseteq \mathcal{J}^*_{n,\,d\geq k}$ and all inclusions are strict for $n \geq k+2$. In the present paper, the asymptotics of the number $|\mathcal{J}^*_{n,\,d\geq k}|$ is found. As a consequence, it is proved that these three classes of graphs have the same asymptotic cardinality. Used methods of graph theory and method of mathematical induction led to a fairly simple proof of calculating the asymptotics of the number $|\mathcal{J}_{n,\,d=k}|$.

Theorem. *Let $k \geq 3$ and $0 < \varepsilon < 1$. Then there is a constant $c_k > 0$ independent of n such that for any $n \in \mathbb{N}$ the following inequalities hold:*

$$2^{\binom{n}{2}}\xi_{n,k}\,(1-\varepsilon_{n,k}) \leq |\mathcal{J}_{n,\,d=k}| \leq |\mathcal{J}_{n,\,d\geq k}| \leq |\mathcal{J}^*_{n,\,d\geq k}| \leq 2^{\binom{n}{2}}\xi_{n,k}\,(1+\varepsilon_{n,k}),$$

$$\textit{where}\ \xi_{n,k} = q_k\,(n)_{k-1}\left(\frac{3}{2^{k-1}}\right)^{n-k+1},\ \varepsilon_{n,k} = c_k\left(\frac{5+\epsilon}{6}\right)^n = o(1),$$

$$q_k = \frac{1}{2}(k-2)\,2^{-\binom{k-1}{2}},\ (n)_k = n(n-1)\cdots(n-k+1).$$

Note that the asymptotic approximation for the number $|\mathcal{J}_{n,\,d=k}|$ obtained in our theorem is more precise than in [4] if $\varepsilon \in (0, \frac{2}{5})$. Furthermore, the constant c_k is indicated explicitly.

Corollary 1. *Let $k \geq 3$. Then the following asymptotic equalities hold:*

$$|\mathcal{J}_{n,\,d=k}| \sim |\mathcal{J}_{n,\,d\geq k}| \sim |\mathcal{J}^*_{n,\,d\geq k}| \sim 2^{\binom{n}{2}}\xi_{n,k}.$$

Corollary 2. *Almost all graphs of a fixed diameter $k \geq 3$ have a unique pair of diametrical vertices, but almost all graphs of the diameter 2 have more than one pair of such vertices.*

Research is supported by the Russian Foundation for Basic Research (project no. 140100507).

References

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