New infinite family of Cameron-Liebler line classes

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based on joint work with Alexander Gavrilyuk,
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Projective geometry

Let $V = GF(q)^{n+1}$.  

$PG(n, q)$ — $n$-dim. projective space over $GF(q)$ 

$\sim : x \sim y \ (x, y \in V \setminus \{0\}) \iff \exists \alpha \in GF(q) : x = \alpha y$ 

$PG(n, q) = (V \setminus \{0\})/\sim$

- point of $PG(n, q)$
  1-dim. vector subspace of $V$

- line of $PG(n, q)$
  2-dim. vector subspace of $V$

- ... 

- hyperplane 
  $n$-dim. vector subspace of $V$

- spread — a line set partitioning the points of $PG(n, q)$
Projective geometry

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- spread — a line set partitioning the points of $PG(n, q)$
Projective geometry $PG(3, q)$

- \( \exists ! \) line through \( \forall \) pair of points with exactly \( q + 1 \) points on a line, while \( \forall \) pair of lines has at most one point in common

- a line belongs to exactly \( q + 1 \) planes, and \( \exists ! \) line in the intersection of \( \forall \) pair of planes

- $PG(3, q)$ always contains a spread (a lot of them)
Projective geometry $PG(3, q)$

- ∃! line through ∀ pair of points with exactly $q + 1$ points on a line, while ∀ pair of lines has at most one point in common

- a line belongs to exactly $q + 1$ planes, and ∃! line in the intersection of ∀ pair of planes

- $PG(3, q)$ always contains a spread (a lot of them)
A Cameron-Liebler line class $\mathcal{L}$ is a set of lines of $PG(3, q)$ such that

$\exists$ a number $x$: for $\forall$ spread $S$

$$|\mathcal{L} \cap S| = x$$

The number $x$ is called the parameter of $\mathcal{L}$.

One can show that $|\mathcal{L}| = x(q^2 + q + 1)$.

A line class $\overline{\mathcal{L}}$ complement to $\mathcal{L}$ is also a Cameron – Liebler line class with $x(\overline{\mathcal{L}}) = q^2 + 1 - x(\mathcal{L}) \Rightarrow$ w.l.o.g. $x \leq \frac{q^2+1}{2}$. 
Cameron-Liebler line class: definition

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Cameron-Liebler line class: examples

$\text{Star}(P)$

$x = 1$

$\text{Line}(\pi)$

$x = 2$

$\text{Star}(P) \cup \text{Line}(\pi)$
Cameron-Liebler conjecture and motivation

Conjecture (Cameron, Liebler, 1982)

The only Cameron-Liebler line classes are those shown above (i.e., \( x \not\in \{3, \ldots, \frac{q^2+1}{2}\} \)).

- This problem is related to the problem of classification of the collineation groups of \( PG(n, q) \) with the same number of orbits on points and lines.
  - Solved by Bamberg and Penttila.

- The Cameron-Liebler line classes give rise to completely regular codes of strength 1 in the Grassmann graphs \( J_q(4, 2) \).

- The Cameron-Liebler line classes give rise to some point sets in \( PG(5, q) \) with two intersection numbers with respect to planes \( \Rightarrow \) projective 2-weight codes \( \Rightarrow \) strongly regular graphs.
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The conjecture was disproved by Bruen and Drudge (1999). They constructed an infinite family of Cameron-Liebler line classes with parameter $x = \frac{q^2+1}{2}$ for all odd $q$. 
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Counterexamples

- $x = 7$ in $PG(3, 4)$.

   (Govaerts, Penttila’05)

- In 2011 M. Rodgers constructed new Cameron–Liebler line classes for many odd values of $q$ ($q < 200$) satisfying $q \equiv 1 \text{ mod } 4$ and $q \equiv 1 \text{ mod } 3$, having parameter $x = \frac{1}{2}(q^2 - 1)$.

   These new examples are made up of a union of orbits of a cyclic collineation group having order $q^2 + q + 1$.

   Rodgers, 2011.

- $x = 10$ in $PG(3, 5)$.

   G. and Metsch, 2013.

- a new infinite family in $PG(3, q)$, $q \equiv 5$ or $9 \text{ mod } 4$,

  $x = (q^2 - 1)/2 \Rightarrow x = (q^2 + 1)/2$.

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Bound on $x$

It seems that the right question is about the lower bound for $x$.

- $x \neq 3, 4$ if $q \geq 5$.  
  (Penttila’91)

- $x \notin \{3, \ldots, \sqrt{q}\}$.  
  (Bruen, Drudge’98)

- $x \notin \{3, \ldots, e(q)\}$ where $q + 1 + e(q)$ is the size of the smallest non-trivial blocking set in $PG(2, q)$.  
  (Drudge’99)

- $x \notin \{3, \ldots, q\}$.  
  (Metsch’10)

- $x > cq^{4/3}$ (with some constant $c$).  
  (Metsch’14)

- about a half of values from $\{3, \ldots, \frac{q^2+1}{2}\}$ cannot be the parameters of Cameron-Liebler line classes.  
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Elliptic quadric, $q$ odd

The elliptic quadric $Q$ is the set of zeros of the equation

$$Q(x_0, x_1, x_2, x_3) = -ax_0^2 + x_1^2 + x_2x_3 = 0,$$

where $a$ is any non-square in $GF(q)$.

- $q^2 + 1$ points;
- any line intersects $Q$ in at most 2 points;
- there exists a unique tangent plane to any point of $Q$;
Bruen-Drudge family of C.-L. line classes

The set of $q + 1$ tangents to a point of $Q$ can be partitioned into two groups, depending on whether there exists a point $P$ on a tangent with $Q(P)$ a square or non-square in $GF(q)$. 
Bruen-Drudge family of C.-L. line classes

A Cameron-Liebler line class of the Bruen-Drudge family consists of all secants to $Q$, and a half of tangents that all correspond to squares (or non-squares).
Lemma (Penttila)
Let $\mathcal{L}$ be a Cameron-Liebler line class such that there exists an incident point-plane pair $(P, \pi)$ satisfying the following conditions:

- $\text{Line}(\pi) \setminus \text{Star}(P) \not\subseteq \mathcal{L}$,
- $\text{Star}(P) \setminus \text{Line}(\pi) \subseteq \mathcal{L}$.

Then

$$\mathcal{L} \cup (\text{Line}(\pi) \setminus \text{Star}(P)) \setminus (\text{Star}(P) \setminus \text{Line}(\pi))$$

is a Cameron-Liebler line class $\mathcal{L}'$ with the same parameter.
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Switching: proof

\[ \mathcal{L}' := \mathcal{L} \cup (\text{Line}(\pi) \setminus \text{Star}(P)) \setminus (\text{Star}(P) \setminus \text{Line}(\pi)) \]

Proof:
For any spread \( S \):
either

\[ S \text{ contains a line on } P \text{ and in } \pi \Rightarrow S \cap \mathcal{L}' = S \cap \mathcal{L}, \]

or

\[ S \text{ contains a line } \ell \in \pi, P \notin \ell, \text{ and a line } m \ni P, m \notin \pi \Rightarrow S' \cap \mathcal{L}' = (S' \cap \mathcal{L}) \cup \{m\} \setminus \{\ell\}. \]

Thus,

\[ |S \cap \mathcal{L}'| = |S \cap \mathcal{L}| = x. \]
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Switching: application

There exists an incident point-plane pair \((P, \pi)\) satisfying the following conditions:

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- \(\text{Star}(P) \setminus \text{Line}(\pi) \subseteq \mathcal{L}\).

Switching with respect to a point of \(Q\) and its tangent plane produces a new Cameron-Liebler line class with \(x = \frac{q^2+1}{2}\).
Switching ⇒ \( x = \frac{q^2+1}{2} \)

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Let \( \mathcal{L} \) be a Cameron-Liebler line class such that there exists an incident point-plane pair \((P, \pi)\) satisfying the following conditions:

- \( \line(\pi) \setminus \star(P) \not\subseteq \mathcal{L} \),
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Then the parameter \( x \) of \( \mathcal{L} \) is equal to \( \frac{q^2+1}{2} \).

In \( PG(3, 5) \) with \( x = \frac{q^2+1}{2} \):

- the Bruen-Drudge example;
- the switched Bruen-Drudge example;
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In \( PG(3, 5) \) with \( x = \frac{q^2+1}{2} \):

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- the Rodgers example;
Let $l$ be a line of $PG(3, q)$, $\mathcal{L}$ a Cameron – Liebler line class. Consider all the points $P_i, i = 1, \ldots, q + 1$ that are on $l$, and all the planes $\pi_j, j = 1, \ldots, q + 1$ that contain $l$.

Define a square matrix $T$ of order $q + 1$ whose $(i, j)$-element is $|\text{pencil}(P_i, \pi_j) \cap \mathcal{L} \setminus \{l\}|$.

We will call such matrix a pattern w.r.t. $l$. 
Properties of patterns

Let $T := (t_{ij})$ be a pattern w.r.t. a line $l$, and define

$$\chi := \begin{cases} 
0 & \text{if } l \notin \mathcal{L}, \\
1 & \text{if } l \in \mathcal{L}, 
\end{cases}$$

Then the following hold:

- $t_{ij} \in \mathbb{N}$, $0 \leq t_{ij} \leq q$ for all $i, j \in \{1, \ldots, q + 1\}$;
- $\sum_{i,j=1}^{q+1} t_{ij} = x(q + 1) + \chi(q^2 - 1)$;
- $\sum_{j=1}^{q+1} t_{kj} + \sum_{i=1}^{q+1} t_{il} = x + (q + 1)(t_{kl} + \chi), \forall k, l$;
- $\sum_{i,j=1}^{q+1} t_{ij}^2 = (x - \chi)^2 + q(x - \chi) + \chi q^2(q + 1)$. 
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- $t_{ij} \in \mathbb{N}$, $0 \leq t_{ij} \leq q$ for all $i, j \in \{1, \ldots, q + 1\}$ ;
- $\sum_{i,j=1}^{q+1} t_{ij} = x(q + 1) + \chi(q^2 - 1)$ ;
- $\sum_{j=1}^{q+1} t_{kj} + \sum_{i=1}^{q+1} t_{il} = x + (q + 1)(t_{kl} + \chi), \forall k, l$ ;
- $\sum_{i,j=1}^{q+1} t_{ij}^2 = (x - \chi)^2 + q(x - \chi) + \chi q^2(q + 1)$. 

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- With Ilia Matkin, we re-discovered this family during his visit at USTC (May of 2016).

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