On the Deza graphs with disconnected second neighbourhood

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Edge and coedge regular graphs

We consider undirected graphs without loops and multiple edges.

For a graph Γ and its an arbitrary vertex x define the i-th neighbourhood $N_i(x) := \{y \mid y \in V(\Gamma), d(x,y) = i\}$ of the vertex x.

For a graph Γ and its an arbitrary vertex x define neighbourhood $N(x) \equiv N_1(x)$ of the vertex x.

A graph Γ is called regular of valency k, if for all $x \in \Gamma \quad |N(x)| = k$.

For a graph Γ and its pair x, y of vertices denote $N(x) \cap N(y)$ by $N(x, y)$.

A graph Γ is called edge regular with parameters (v, k, λ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$ the following holds

$$|N(x,y)| = \begin{cases} k, & \text{if } x = y; \\ \lambda, & \text{if } x \sim y. \end{cases}$$

A graph Γ is called coedge regular with parameters (v, k, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$ the following holds

$$|N(x,y)| = \begin{cases} k, & \text{if } x = y; \\ \mu, & \text{if } x \neq y \text{ and } x \not\sim y. \end{cases}$$

Strongly regular and Deza graphs

A graph Γ is called strongly regular (SRG) with parameters (v, k, λ, μ), if Γ has v vertices, and for any pair of vertices $x, y \in \Gamma$ the following holds

$$|N(x,y)| = \begin{cases} k, & \text{if } x = y; \\ \lambda, & \text{if } x \sim y; \\ \mu, & \text{if } x \neq y \text{ and } x \nsim y. \end{cases}$$

A graph is called a Deza graph with parameters (v, k, b, a) (usually $a \leq b$), if it has v vertices, and for any pair of vertices x, y the following holds

$$|N(x,y)| = \begin{cases} k, & \text{if } x = y; \\ a \text{ or } b, & \text{if } x \neq y. \end{cases}$$

Deza graphs are natural generalization of strongly regular graphs.

A Deza graph is called a strictly Deza graph, if it has diameter 2, and is not SRG.

Preliminary results and problem

Problem 1

Classify strongly regular graphs which contain a vertex with disconnected second neighbourhood.

Theorem (Gardiner A.D., Godsil C.D., Hensel A.D., Royle G.F. 1992)

Let Γ *be a strongly regular graph. If there is* $u \in V(\Gamma)$*, such that* $N_2(u)$ *is disconnected, then* $N_2(u)$ *contains no edges and* Γ *is a complete multipartite graph with* $s \geq 2$ *parts of the same size* $t > 2$*.*

Problem 2

Classify strictly Deza graphs which contain a vertex with disconnected second neighbourhood.

In this work we consider "extremal" cases of coedge regular and edge regular strictly Deza graphs and also the general case of Deza graphs such that the second neighbourhood of each vertex is disconnected.

Composition of graphs

Definition

Let $\Gamma_1 = (V_1, E_1)$ *and* $\Gamma_2 = (V_2, E_2)$ *be graphs. The composition* $\Gamma_1[\Gamma_2]$ *of graphs* Γ_1 *and* Γ_2 *is the graph with vertex set* $V_1 \times V_2$ *and the adjacency rule*

$$(u_1, v_1) \sim (u_2, v_2) \Leftrightarrow u_1 \sim u_2 \ OR \ (u_1 = u_2 \ AND \ v_1 \sim v_2)$$

Construction 1 of Deza graphs

Proposition

Let Γ *be a complete multipartite graph with* $s \geq 2$ *parts of the same size* $t > 2$*. Denote* $D(t,s) := \Gamma[K_2]$ *then*

1. $D(t,s)$ *is a strictly Deza graph with parameters*

$$(2ts,\ 2t(s-1)+1,\ 2t(s-1),\ 2t(s-2)+2);$$

2. $D(t,s)$ *is a coedge regular graph;*
3. $\forall x \in D(t,s)$ *the second neigbourhood of* x *is a disconnected graph.*

Remark

$D(t,s)$ *is a vertex transitive graph.*

Construction 2 of Deza graphs

Proposition

Let Γ_1 be an (n,k,λ,μ) strongly regular graph with $\lambda=\mu$ and Γ_2 be an n'-coclique, where $n'\geq 2$, then

1. $\Gamma_1[\Gamma_2]$ *is a strictly Deza graph with parameters*
$$(nn', kn', kn', \lambda n');$$
2. $\Gamma_1[\Gamma_2]$ *is an edge regular graph;*
3. $\forall x\in\Gamma_1[\Gamma_2]$ *the second neigbourhood of x is a disconnected graph.*

There is a not vertex transitive SRG with parameters $(45,12,3,3)$. So, Contruction 2 gives the infinite series of not vertex transitive edge regular strictly Deza graphs.

The problem of existence of SRG with $\lambda=\mu$ is open in general case.

Example: $(153,96,60,60)$ is the smallest(w.r.t. number of vertices) set of parameters of SRG with $\lambda=\mu$ for which the existence of a graph is unknown.

Our results

Theorem 1 (Common case)

Let Γ be a strictly Deza graph. If the second neighbourhood of each vertex is disconnected then Γ is either edge-regular or coedge regular.

Theorem 2 (Coedge regular case)

Let Γ be a coedge regular strictly Deza graph. If there is $u \in V(\Gamma)$, such that $\Gamma_2(u)$ is disconnected, then $\Gamma \cong D(t, s)$ with appropriate values of parameters.

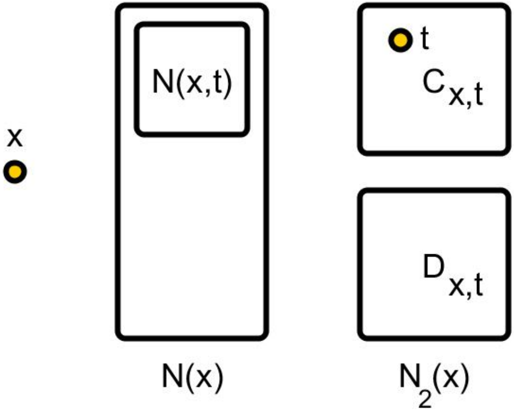
Theorem 3 (Edge regular case)

Let Γ be an edge regular strictly Deza graph. If there is $u \in V(\Gamma)$, such that $\Gamma_2(u)$ is disconnected, then $\Gamma \cong \Gamma_1[\Gamma_2]$ where Γ_1 is a strongly regular graph with $\lambda = \mu$ and Γ_2 is a coclique of size $s \geq 2$.

Proof scheme of theorem 1

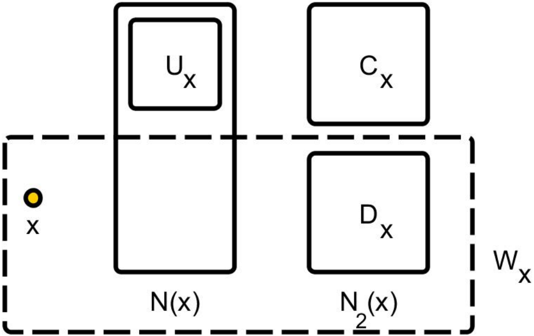
Suppose the contrary that graph is not edge regular and coedge regular. Then we obtain some properties of graph in this assumption. And finally we get a contradiction.

- Consider vertex x and vertex $t \in N_2(x)$ which has a common neighbours with x. Denote the connected component of $N_2(x)$ which contains t by $C_{x,t}$ and $N_2(x) \setminus C_{x,t}$ denote by $D_{x,t}$.
- Each vertex from $N(x,t)$ has a common neighbours with x.
- Each vertex from $D_{x,t}$ has b common neighbours with x.
- Each vertex from $C_{x,t}$ has a common neighbours with x.


x
N(x,t)
N(x)
t
C x,t
D x,t
N 2(x)

Proof scheme of theorem 1

- $C_{x,t}$ and $D_{x,t}$ are independent of the vertex t, and we denote its by C_x and D_x. Denote $\bigcup\limits_{t \in C_{x,t}} N(x,t)$ by U_x.
- Each vertex from U_x is adjacent with each vertex from D_x.
- Each vertex from U_x is adjacent with each vertex from $N(x) \setminus U_x$.
- Each vertex from U_x has a common neighbours with x.


U_x
C_x
x
D_x
W_x
N(x)
N_2(x)

Proof scheme of theorem 1

- Denote subgraph induced by $\{x\} \cup D_x \cup (N(x) \setminus U_x)$ by W_x. Each vertex from W_x has b common nieghbours with each other vertex from W_x and has a common neighbours with each vertex from $\Gamma \setminus W_x$.
- W_x is strongly regular graph with parameters $(\beta + 1, k - |U_x|, b - |U_x|, b - |U_x|)$.
- Let R be the binary relation on the vertex set of the graph Γ defined by the following rule "either to coincide or to have b common neighbours". Then R is an equivalence relation.
- Consider the graph on the set of equivalent classes such that two classes are adjacent iff in Γ there is edges between these classes. This graph is strongly regular and we get finally contradiction with properties of strongly regular graphs.

Open problems

Open problem 1

Are there strictly Deza graphs which have vertices with connected and disconnected second neighbourhoods, a not edge regular and a not coedge regular?

If the previous problem has positive solution, then

Open problem 2

Classify strictly Deza graphs which have vertices with connected and disconnected second neighbourhoods, a not edge regular and a not coedge regular.

Thank you for your attention!