On the realizability of a finite graph whose complement is triangle-free as the Gruenberg–Kegel graph of a finite group

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By “group” we mean “a finite group” and by “graph” “an undirected graph without loops and multiple edges”. Let G be a group. Denote by $\pi(G)$ the set of all prime divisors of the order of G and by $\omega(G)$ the *spectrum* of G, i.e. the set of all its element orders. The set $\omega(G)$ defines the *Gruenberg–Kegel graph* (or the *prime graph*) $\Gamma(G)$ of G; in this graph the vertex set is $\pi(G)$ and different vertices p and q are adjacent if and only if $pq \in \omega(G)$.

We say that a graph Γ with $|\pi(G)|$ vertices *is realizable as the Gruenberg–Kegel graph of a group* G if there exists a labeling of the vertices of Γ by distinct primes from $\pi(G)$ such that the labeled graph is equal to $\Gamma(G)$. In [1] it was proved, a graph is realizable as the Gruenberg–Kegel graph of a solvable group if and only if its complement is 3-colorable and triangle free. In [2] it was proved, the Gruenberg–Kegel graph of an almost simple group is isomorphic to the Gruenberg–Kegel graph of an appropriate solvable group if and only if its complement is triangle free. The following question arises.

Question. Is there a graph without 3-cocliques, whose complement is not 3-colorable, but which is isomorphic to the Gruenberg–Kegel graph of an appropriate non-solvable group? In the other words, is there a graph which is realizable as the Gruenberg–Kegel graph of an appropriate non-solvable group, but is not realizable as the Gruenberg–Kegel graph of any solvable group?

The Grotzsch graph is the smallest example of triangle-free graph which is not 3-colorable (see Fig. 1).

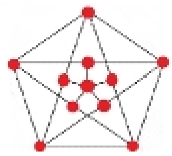

Fig. 1: Grotzsch graph

We prove the following theorem.

Theorem. *The complement of the Grotzsch graph is not realizable as the Gruenberg–Kegel graph of a group.*

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