# The four color theorem and Thompson's *F and links*

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# Thompson's F

$\boldsymbol{Def(Thompson's\ F)}$

*Condition Q*

- $\varphi$: [0,1] → [0,1] is piecewise linear homeomorphism
- $\varphi$ is differentiable except at finitely $\frac{b}{2^a}$ form numbers $(a, b \in \mathbb{Z})$
- on differentiable interval of $\varphi$, the derivatives are powers of 2

$F$ ≔ $\{\varphi \mid \varphi$ meets condition $Q\}$ is a group by composition of maps.

$$F \cong \langle A, B \mid [AB^{-1}, A^{-1}BA] = [AB^{-1}, A^{-2}BA^2] = \mathrm{id} \rangle$$

with $[x, y] = xyx^{-1}y^{-1}$

Cannon, J.W., Floyd, W.J., Parry, W.R.: Introductory notes on Richard Thompson's groups. Enseign. Math. (2) 42(3–4), 215–256 (1996)

# Four color theorem

**Four color theorem**

Every planar graph has a face 4-coloring.

$\Leftrightarrow$

Let $F$ be Thompson's $F$. $\forall f \in F, f$ is colorable.

By Bowlin and Brin, 2013

# Binary trees

***Def* (Binary tree)**

$\{0,1\}^* := \{\textit{finite words in the alphabets } 0 \textit{ and } 1\} \cup \{\emptyset\}$

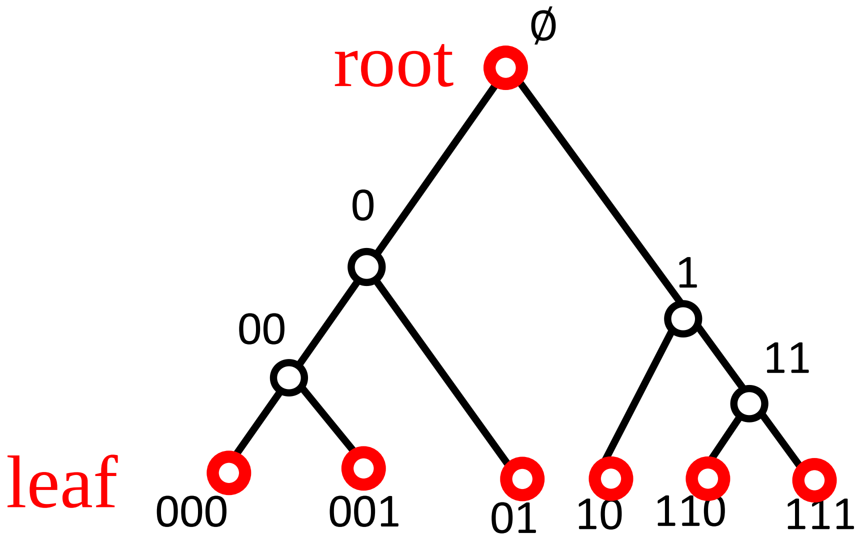
If a finite set $G$ satisfies these conditions as follows

*1.* $G \subset \{0,1\}^*, \emptyset \in G$,

*2.* $\forall w \in G, (w0 \in G \wedge w1 \in G) \vee (w0 \notin G \wedge w1 \notin G)$,

*3.* $w0 \in G \vee w1 \in G \Rightarrow w \in G$,

then we say that $G$ is a **binary tree**.




Ex: $G = \{\emptyset, 0, 1, 00, 01, 000, 001, 10, 11, 110, 111\}$

# Binary trees

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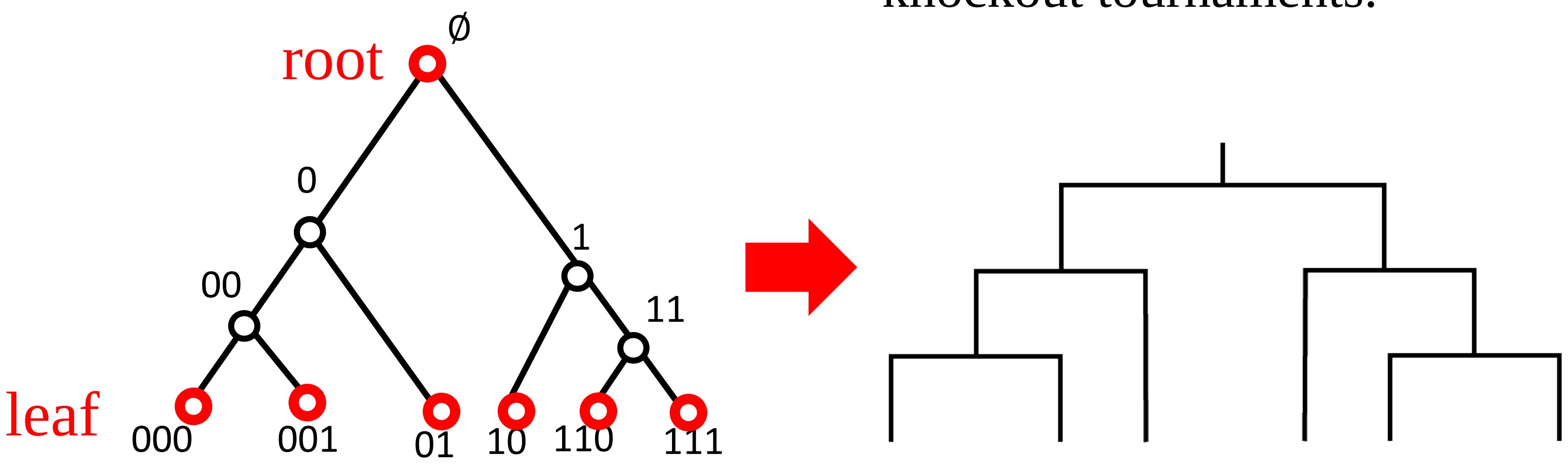
then we say that $G$ is a **binary tree**.

We should regard binary trees as knockout tournaments.

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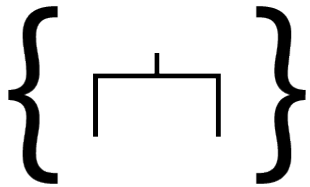
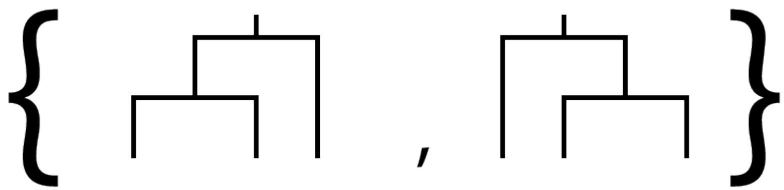
# Binary trees

$$T_n := \{binary\ trees\ having\ n\ leaves\}$$

$T_1 =$ 

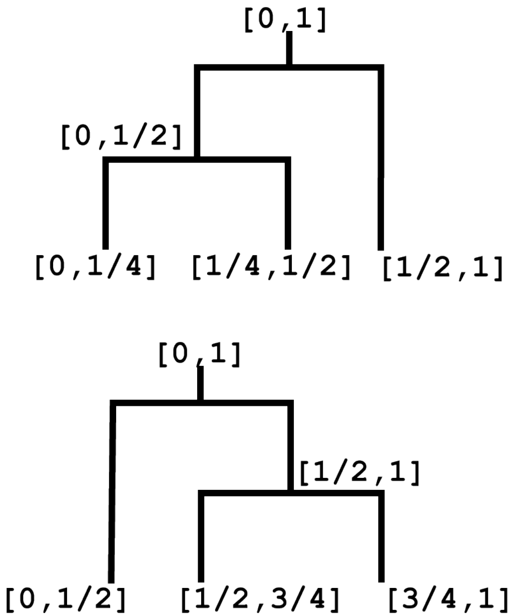
$T_2 =$ 

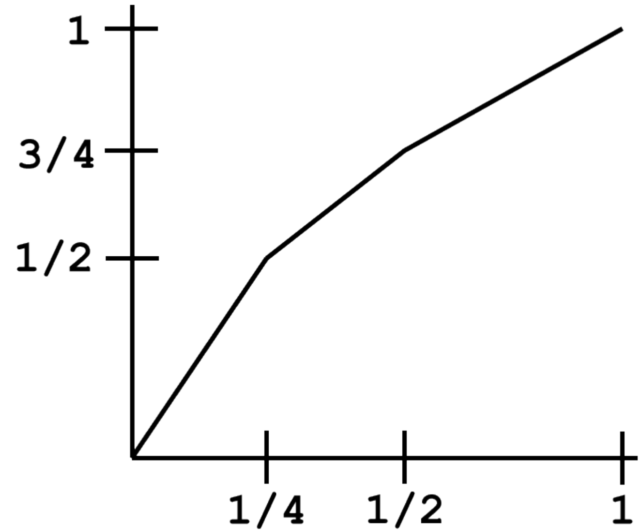
$T_3 =$ 

$T_4 =$ 

# Thompson's F

We can get a map $\varphi$: [0,1] → [0,1] from a pair of binary trees.

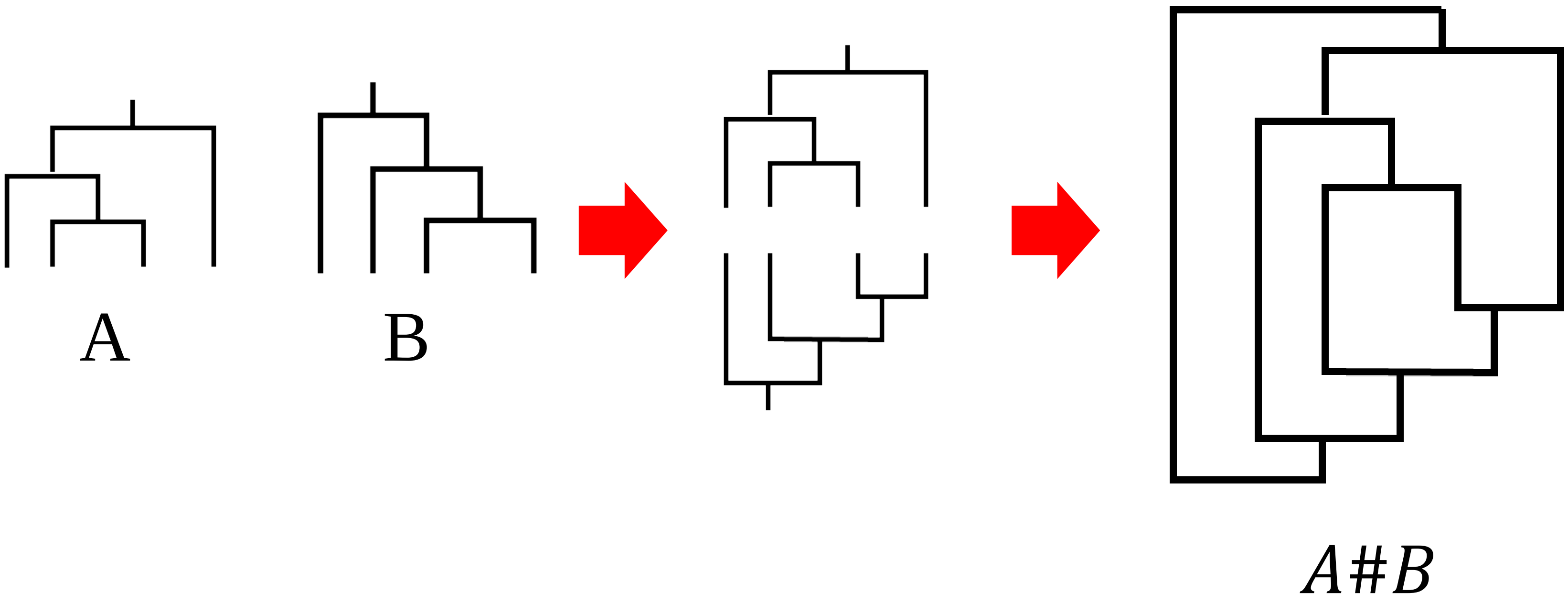



$$\varphi(x) = \begin{cases} 2x & \text{if } x \in [0, 1/4] \\ x + 1/4 & \text{if } x \in [1/4, 1/2] \\ x/2 + 1/2 & \text{if } x \in [1/2, 1] \end{cases} \quad \in F$$

# A#B

$\forall A, B \in T_n$, we get 3 regular graph when connect $A$ and $B$.



# Thompson's $F$

$\boldsymbol{Def}$(Reduced pair)
Let $A, B \in T_n$. If $A\#B$ has no $C_2$, then we say the pair $(A, B)$ is ***reduced.***

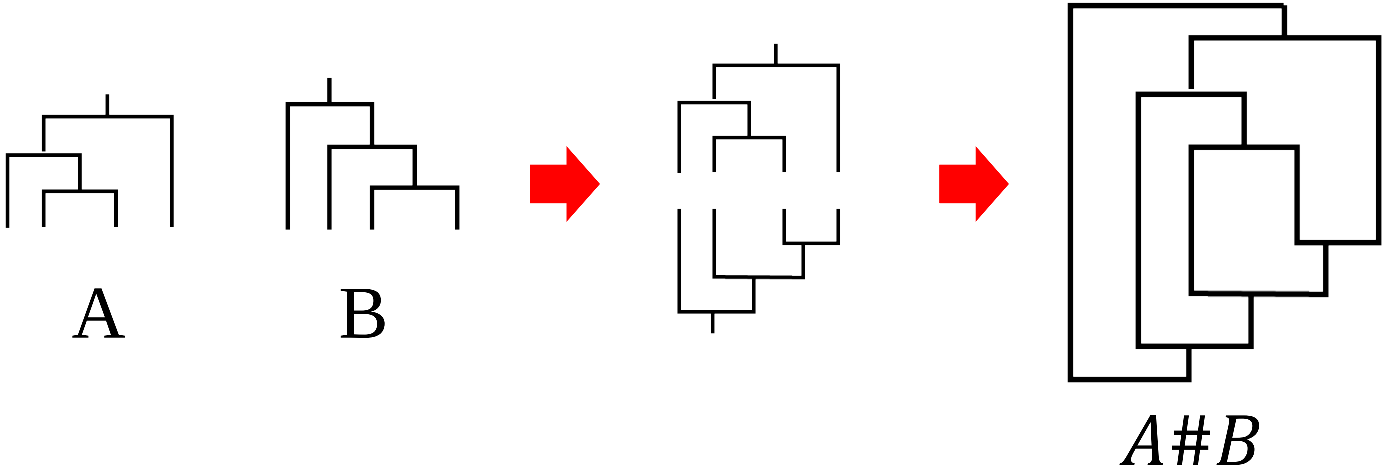
$$r(T_n^2) \coloneqq \{(A, B) \in T_n \times T_n \mid (A, B)\ \mathit{is\ reduced}\}$$

$\boldsymbol{Theorem}$(Bowlin, Brin, 2013)

Let $F$ be Thompson's $F$. There exists a bijection

$$g\colon F \longrightarrow \bigsqcup_{n \in \mathbb{N}} r(T_n^2)\,.$$

# Colorable




***Def***

Let $f \in F$ and $g(f) = (A, B)$.

If $A\#B$ has edge 3-coloring, we say $f$ is **colorable.**

# Four color theorem

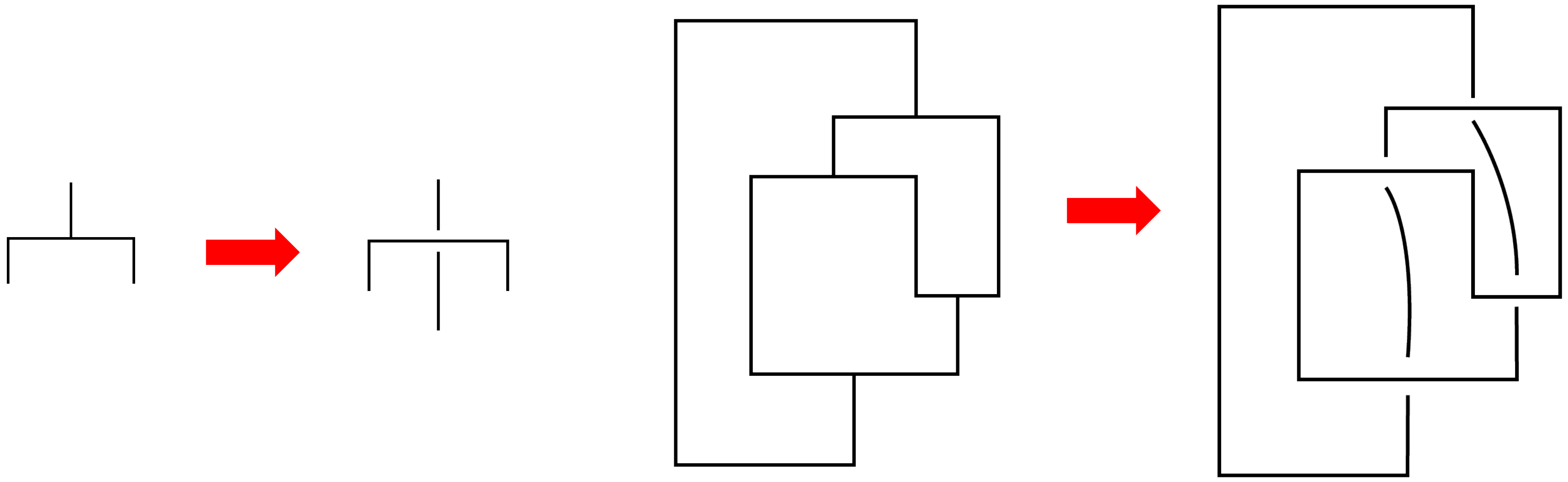
**Four color theorem**

Every planar graph has a face 4-coloring.

$$\Leftrightarrow$$

Let $F$ be Thompson's $F$. $\forall f \in F, f$ is colorable.

It is known that we can make a link with a pair of binary trees.



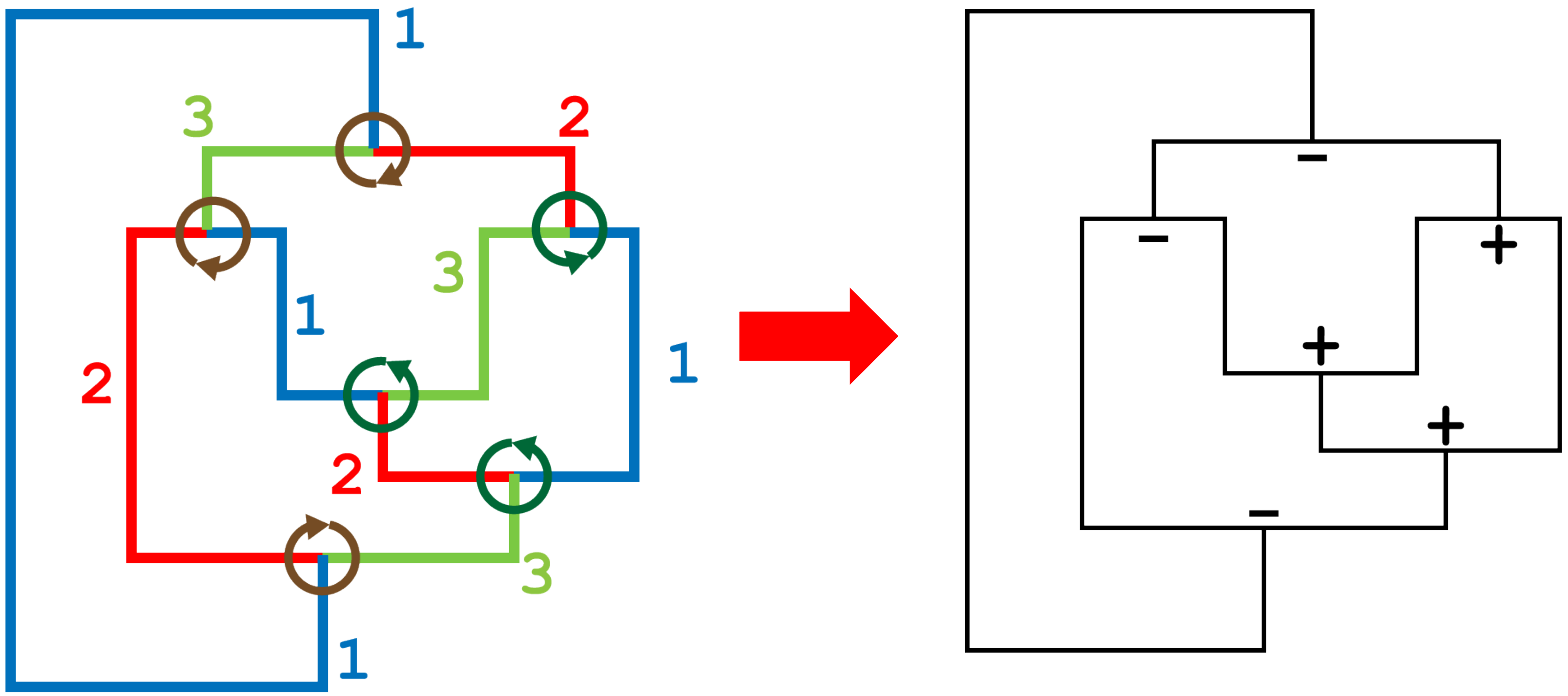
**Question:**

**What will happen if we append information about colorings?**

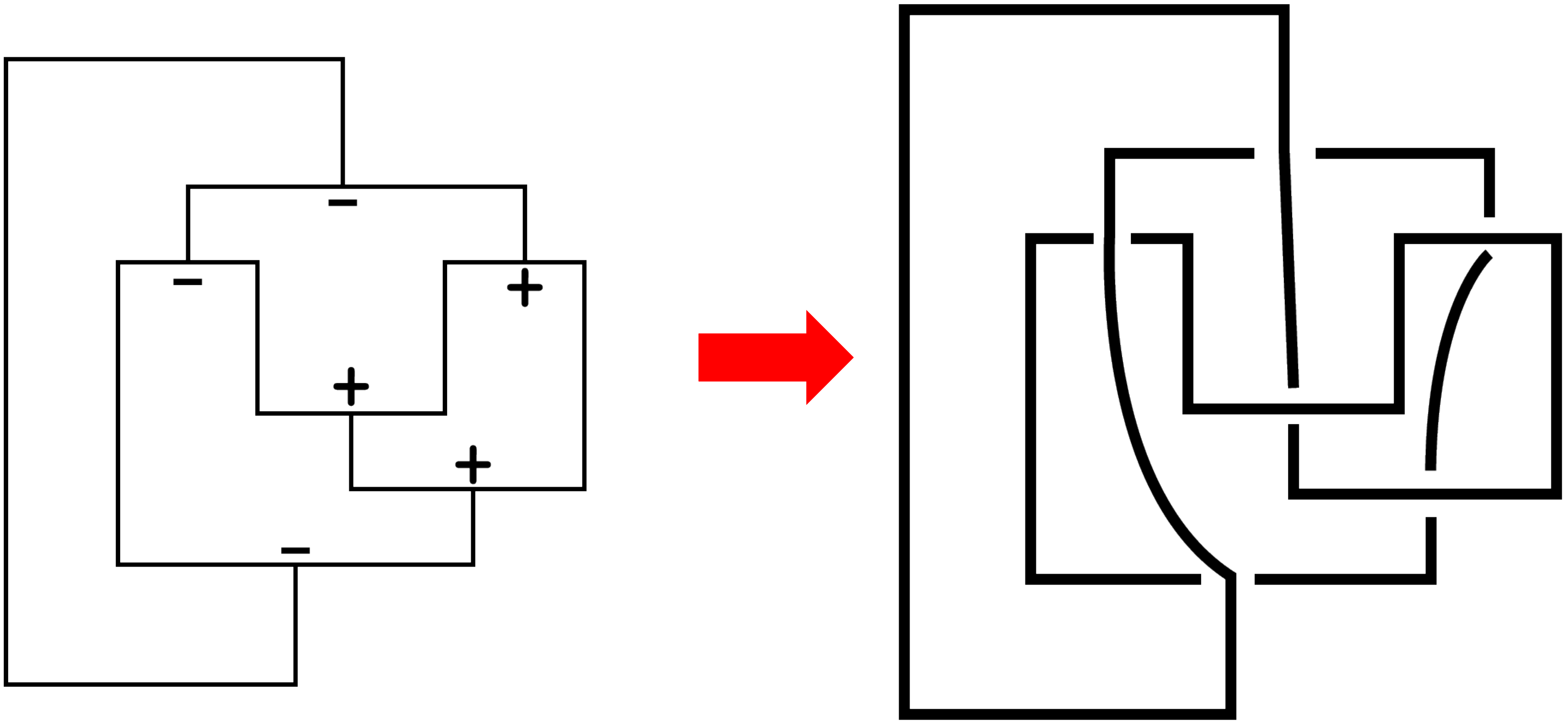
# How to append

How do we append it?

We can attach + or − sign to each vertices with a coloring.



# How to append

# Result

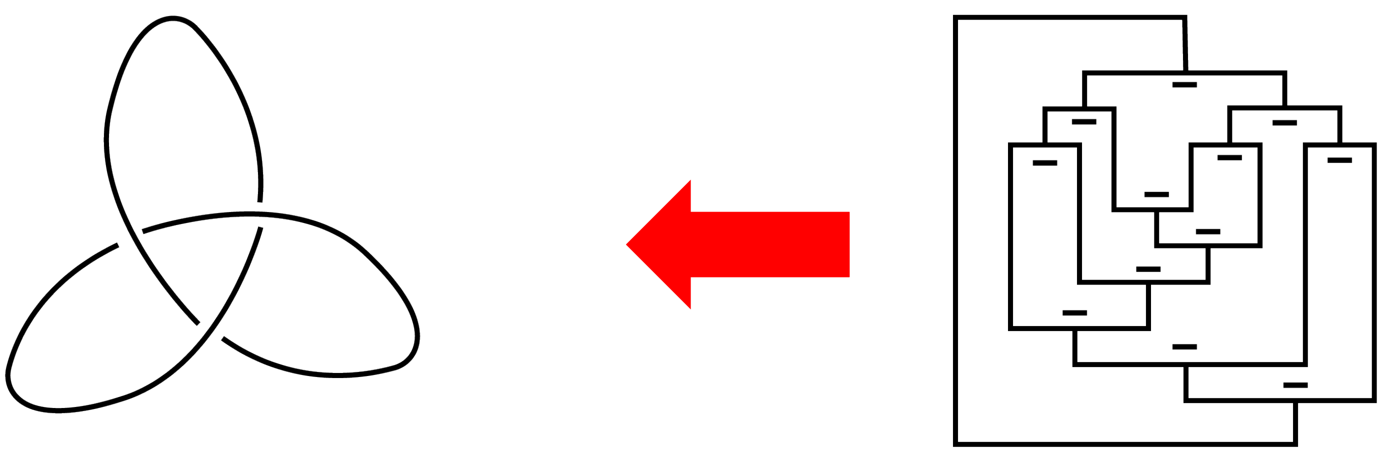
***Def***

$$h:(\cup\, T_n \times T_n) \times \{signs\} \to \{links\}$$

***Theorem***(2 weeks ago)
$h$ is surjective.

Especially, for any **knot** $K$, there exists $f \in F$ and a sign $\sigma$ s.t

$$h(g(f), \sigma) = K.$$



# Result

***Def***

$$h: (\cup T_n \times T_n) \times \{signs\} \to \{links\}$$

***Theorem***(This morning, 7:30)

$h$ is surjective.

Especially, for any **link** $L$, there exists $f \in F$ and a sign $\sigma$ s.t

$$h(g(f), \sigma) = L.$$

# Question

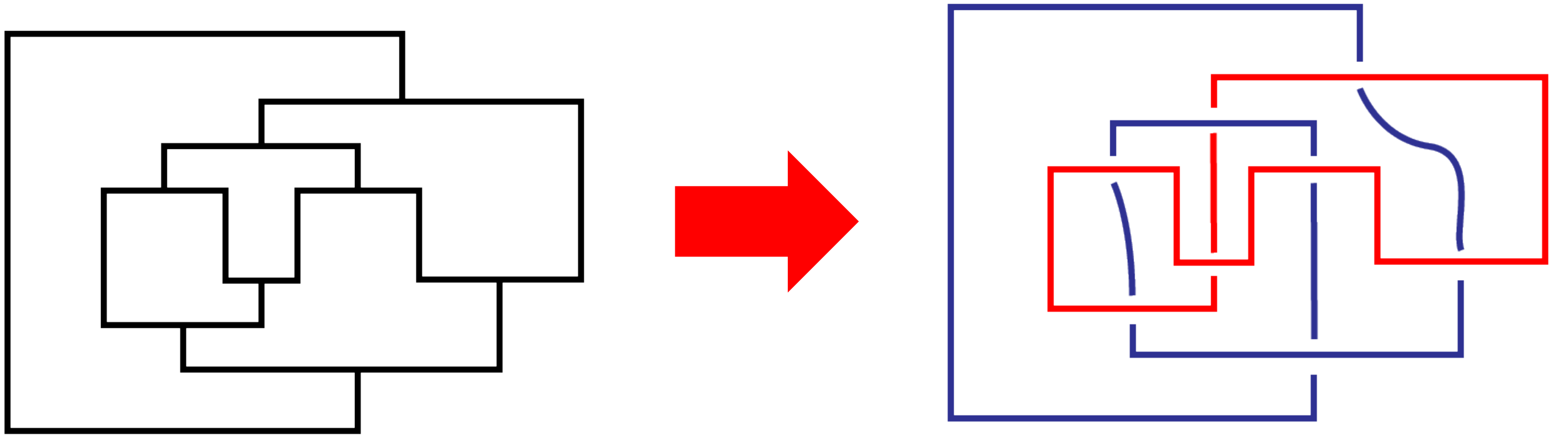
***Lemma***

Let $A, B \in T_n$.

The number of components of $h((A,B),\sigma)$ does not depend on signs.

***Question***

What kind of relationships are there between the number of components of a link and the element of Thompson's $F$?



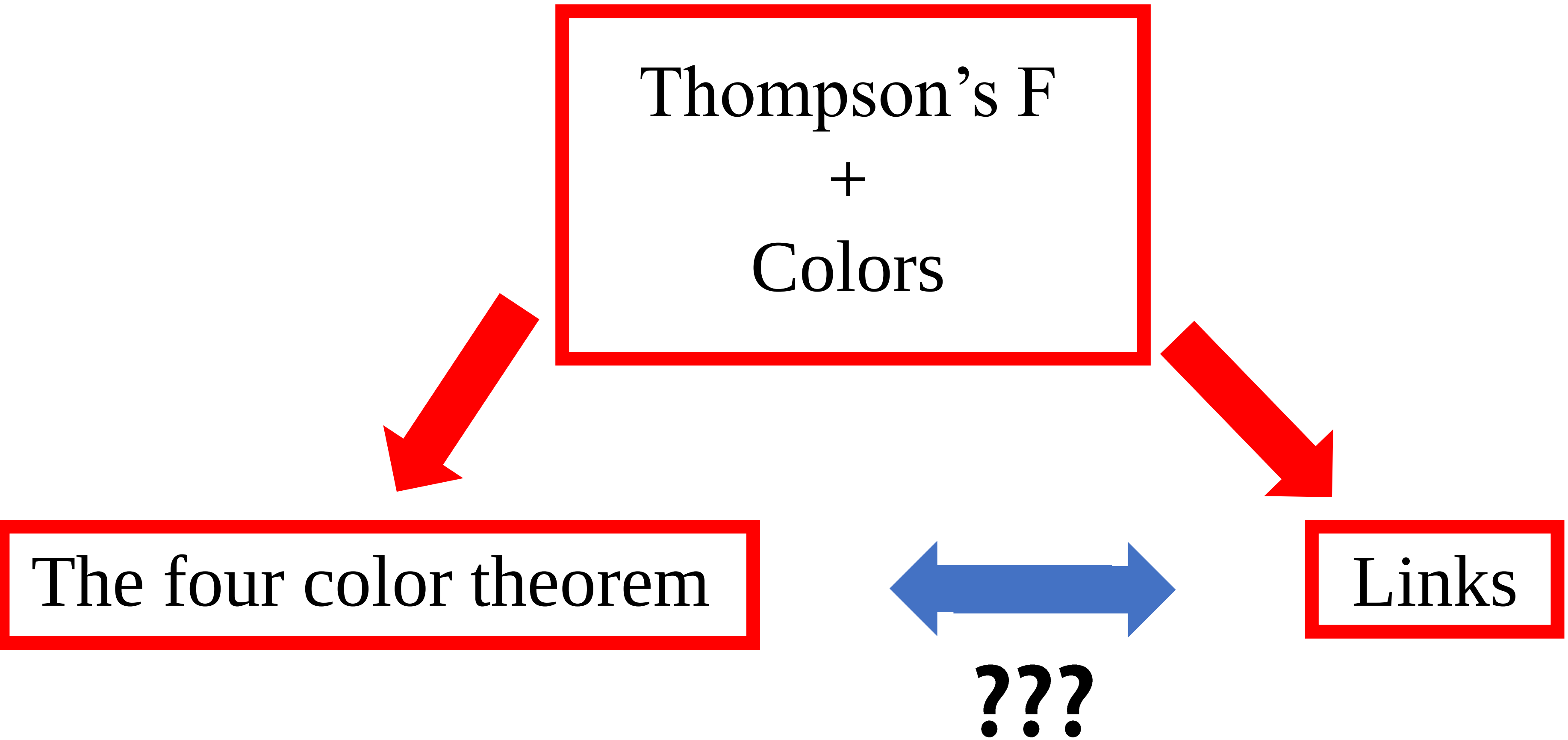
# Question

Thompson's F
+
Colors

The four color theorem

Links

# Question



## Question

Thompson's F

+

# Thank you for your attention!

The four color theorem

Links

???