Group factorizations, graphs and characters of groups

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1. Group factorizations. Let G be a group and A, B be its subgroups. The group G has a factorization $G = AB$ if every element $g \in G$ can be expressed in the form $g = ab$ with $a \in A, b \in B$. By famous Burnside's p^{α}-lemma (1903) the group G, having the conjugacy class of a prime-power size $p^{\alpha} > 1$ is non-simple. Clearly, in this case G has a factorization of the form: $G = C_G(x)P$, where P is a Sylow p-subgroup. As an immediate consequence, every group of order $p^a q^b$ for prime numbers p, q and natural numbers a, b is soluble. Further investigations due to H. Wielandt, O. Kegel, B. Huppert, N. Itô and others leads to many classical result in this area. For instance, finite group $G = AB$ with nilpotent subgroups A and B is soluble. The last result in this area, not using FSGC, is a theorem of L. Kazarin (1979), solving Shemetkov - Scott conjecture: the group $G = AB$ factorized by subgroups A and B such that A and B have nilpotent subgroups A_0 and B_0 of index at most 2 in the corresponding group, is soluble.

Later (in 1990) M. Liebeck, Sh. Praeger and J. Saxle have classified maximal factorizations of all finite simple groups, using FSGC. However, many simple problems, concerning factorizations, remains open. A short survey of the results in this area could be find in [1]. Some new results were obtained in this century.

Recall that the group X is called π-decomposable, if G is a direct product of its Hall π-subgroup $O_{\pi}(X)$ and a subgroup $O_{\pi'}(X)$ of coprime order. The following (containing classical results due H. Wielandt and O. Kegel) was proved by L. Kazarin, A. Martinez-Pastor and M. D. Perez-Ramos [2] in 2015.

Theorem 1. *Let π be a set of odd primes. If a finite group $G = AB$ is a product of two π-decomposable subgroups A and B, then $O_{\pi}(A)O_{\pi}(B)$ is a subgroup of G.*

As a corollary, we prove that the product $G = AB = AC = BC$ of permutable finite π-decomposable subgroups A, B and C is π-decomposable.

A generalization of some results due to Z. Arad, E. Fisman and E. M. Palchik is also presented in the talk.

Note that there is a natural "geometric"situation, when the factorizations are raised. If G is a transitive permutation group acting on a set Ω and a subgroup $H \leq G$ also acts transitively on Ω, then G has a factorization $G = HK$, where K is a stabilizer of a point $\alpha \in \Omega$.

There is another type of factorizations. They are, so-called, ABA-factorizations. More precisely, let A and B be a subgroups of G. We say that G is an ABA-group, if for every element $g \in G$ there exist $a, a' \in A$ and $b \in B$ such that $g = aba'$. There are many interesting classes of groups possessing non-trivial ABA-factorizations. Among them all finite simple groups of Lie type and alternating groups of permutation of degree $n \geq 5$. It is unknown whether every sporadic simple group possesses non-trivial ABA-factorization. There are some interesting results about such factorizations since first papers of D. Gorenstein and I. M. Herstein. But in general the situation is very complicated. One recent result [3] belongs to B. Amberg and L. Kazarin:

Theorem 2. *Let a finite group $G = ABA$ cyclic subgroup B. If A is abelian or A is nilpotent of odd order and $GCD(|A|, |B|) = 1$, then G is soluble.*

Note that the structure of a nonsoluble ABA-group with abelian subgroups A and B is still unknown.

Of course, every 2-transitive permutation group is an ABA-group for every subgroup B, not contained in A. It seems that such factorizations exists more often if G is a rank 3 permutation group. In each case the authors [4] have find some new approach to this problem based on the properties of involutions.

2. Some arithmetic properties of the characters of groups. It is well-known that the main tool for the proofs of theorems concerning groups with factorizations was character theory. This is clear for Burnside's p^α-lemma. In general, a finite group has a factorization $G = AB$ iff $\langle 1_A^G, 1_B^G\rangle = 1$. Similar criterions exist also for 2-transitive groups and rank 3 permutation groups.

E. P. Wigner has proved (in 1941) very interesting result, concerning finite groups with the following property. Let G be a real finite group all whose all irreducible representations are $T_1, T_2, \dots T_k$. If for any $i, j \le k$ the decomposition $T_i \otimes T_j = \sum c_{ij}^s T_s$ has all coefficients $c_{ij}^s \le 1$, then the following holds:

$$\sum_{g\in G} |C_g(g)|^2 = \sum_{g\in G} \zeta(g)^3.$$

Here $\zeta(g)$ is the number of solutions in G of the equation $x^2 = g$. E.Wigner called groups with this property SR-groups. The solubility of finite SR-groups was proved by L. Kazarin with his students in 2010. One of the results of similar nature obtained in 2011 with B. Amberg, is as follows:

Theorem 3. *Let G be a finite simple group and τ be an arbitrary involution of G. If $|G| > 2|C_G(\tau)|$, then G has a proper subgroup of order at least $|G|^{1/2}$. If $|G| > |C_G(\tau)|^3$, then $|G| < k(G)^3$, where $k(G)$ is a class number of G.*

The behavior of the degrees of irreducible characters is of special interest for many authors. One of the famous computational problems in computational mathematics is the complexity of matrix computation. In celebrated works of Umans with coauthors this is reduced in some sense to estimate of the number $\sum_{\chi\in Irr(G)} \chi(1)^3$ for certain groups G. The cases when the group G has an irreducible character of large degree, is very interesting. One new result was obtained in [5] by L. Kazarin and S. Poiseeva.

Theorem 4. *Let G be a finite group with an irreducible character Θ such that $|G| \le 2\Theta(1)^2$. If G is not a 2-group, then every irreducible character of G is a constituent of Θ^2. If $\Theta(1) = pq$ for some primes p and q, then G has an abelian normal subgroup N of index pq.*

There are many simple groups G with the property $|G| < c\chi(1)^2$ for some irreducible character χ of G and some small constant c. As an example, for $c < 3$ there is the Thompson group Th of order 190373967.

3. Graphs on the sets of primes. There are several types of graphs determined on the prime divisors of the order of a group. Let x be a natural number and $\pi(x)$ be the set of its prime divisors. If X is the set of natural numbers, then $\rho(X) = \cup_{x\in X}\pi(x)$. Denote the graph $\Gamma(X)$ with the set $\rho(X) = V(X)$ of its vertices. Two vertices are adjacent if $pq \mid x$ for some $x \in X$.

Another graph $\Delta(X)$ on the set X is defined as follows. Vertices a and b are adjacent, if the greatest common divisor of a and b is bigger than one.

It seems that the first (after Cayley) graphs in group theory were invented by S. A. Chounikhin in 1938. In an explicit form this was done by L. Kazarin in 1978. Prime graph of Grünberg-Kegel, $GK(G)$, became popular since 1981 after paper by J. S. Williams and later by A. S. Kondratiev in connection of a program of characterization of a simple groups by spectrums. In $GK(G)$ the set X is the set of prime divisors of elements of G. In this case primes p and q are adjacent if there exists in G an element whose order is pq.

Another prime graph $\Gamma_{sol}(G)$ was invented by S. Abe and N. Iiyori. In this case X is the set of a prime divisors of soluble subgroups of G. Two primes p and q are adjacent if there exists a soluble subgroup of G whose order is divisible by pq.

One of recent results for this graphs related to finite simple groups is due B. Amberg and L. Kazarin. Previously S. Abe and N. Iiyori described finite simple groups whose graph $\Gamma_{sol}(G)$ is a clique. Define by $t_s(G)$ the largest number of independent vertices in $\Gamma_{sol}(G)$.

Theorem 4. *Let G be a finite simple group such that $t_s(G) = 2$ (i.e. the dual graph to $\Gamma_{sol}(G)$ has no triangles). Then G is isomorphic to one of the following groups: $L_n^{\pm}(q)(n \le 7), S_4(q), P\Omega_8^+(2), {}^3D_4(2), {}^2F_4(2)', G_2(3), S_6(2), M_{11}, M_{12}, M_{22}, HS, McL, J_2$ or $A_n(n \le 10)$.*

The proof uses two papers by A. V. Vasiliev and E. P. Vdovin. As a corollary we obtain the description of slightly larger class of finite simple groups, than groups having a factorization by two soluble subgroups.

Theorem 5. *Let G be a finite simple group with soluble subgroups A and B. If $\pi(G) = \pi(A) \cup \pi(B)$, then G belongs to the list of groups in the conclusion of Theorem 4.*

The graph $\Gamma_A(G)$ was defined by L. Kazarin, A. Martinez-Pastor and M. D. Perez-Ramos in 2005. This graph is defined on the set of prime divisors of the order of a group G in a following manner. Two vertices p and q in $\pi(G)$ are adjacent if for a Sylow p subgroup P of G the order of a group $N_G(P)/PC_G(P)$ is divisible by q. Of course, the edge (q, p) exists if $|N_G(Q)/QC_G(Q)|$ is divisible by p.

One of the important results concerning these graphs is as follows:

Theorem 6. *Let G be a finite almost simple group. Then the graph $\Gamma_A(G)$ is connected.*

Note that if (p, q) is an edge in $\Gamma_A(G)$, then (p, q) is an edge in $\Gamma_{sol}(G)$, but the graph $\Gamma_A(G)$ of a soluble group could be non-connected. Hence our theorem 6 gives another proof of a theorem by S. Abe and N. Iiyori. Theorem 6 is a main tool for some results in formation theory concerning formation closed uner taking of normalizers of Sylow subgroups.

References

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