Affine connections on three-dimensional homogeneous spaces

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Let $(\overline{G}, M)$ be a three-dimensional homogeneous space, where $\overline{G}$ is a Lie group on the manifold M. We fix an arbitrary point $o \in M$ and denote by $G = \overline{G}_o$ the stationary subgroup of o. It is known that the problem of classification of homogeneous spaces $(\overline{G}, M)$ is equivalent to the classification (up to equivalence) of pairs of Lie groups $(\overline{G}, G)$ such that $G \subset \overline{G}$. In the study of homogeneous spaces it is important to consider not the group $\overline{G}$ itself, but its image in $\mathrm{Diff}(M)$. In other words, it is sufficient to consider only effective actions of $\overline{G}$ on M. Since we are interested only the local equivalence problem, we can assume without loss of generality that both $\overline{G}$ and G are connected. Then we can correspond the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ of Lie algebras to $(\overline{G}, M)$, where $\bar{\mathfrak{g}}$ is the Lie algebra of $\overline{G}$ and $\mathfrak{g}$ is the subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. This pair uniquely determines the local structure of $(\overline{G}, M)$, that is two homogeneous spaces are locally isomorphic if and only if the corresponding pairs of Lie algebras are equivalent. A pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is *effective* if $\mathfrak{g}$ contains no non-zero ideals of $\bar{\mathfrak{g}}$, a homogeneous space $(\overline{G}, M)$ is locally effective if and only if the corresponding pair of Lie algebras is effective. An *isotropic* $\mathfrak{g}$-*module* $\mathfrak{m}$ is the $\mathfrak{g}$-module $\bar{\mathfrak{g}}/\mathfrak{g}$ such that $x.(y+\mathfrak{g}) = [x, y]+\mathfrak{g}$. The corresponding representation $\lambda\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{m})$ is called an *isotropic representation* of $(\bar{\mathfrak{g}}, \mathfrak{g})$. The pair $(\bar{\mathfrak{g}}, \mathfrak{g})$ is said to be *isotropy-faithful* if its isotropic representation is injective. If there exists at least one invariant connection on $(\bar{\mathfrak{g}}, \mathfrak{g})$ then this pair is isotropy-faithful [1].

We divide the solution of the problem of classification all three-dimensional isotropically–faithful pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ into the following parts. We classify (up to isomorphism) faithful three-dimensional $\mathfrak{g}$-modules U, this is equivalent to classifying all subalgebras of $\mathfrak{gl}(3, \mathbb{R})$ viewed up to conjugation. For each obtained $\mathfrak{g}$-module U we classify (up to equivalence) all pairs $(\bar{\mathfrak{g}}, \mathfrak{g})$ such that the $\mathfrak{g}$-modules $\bar{\mathfrak{g}}/\mathfrak{g}$ and U are isomorphic. All of this pairs are described in [2].

Invariant affine connections on $(\overline{G}, M)$ are in one-to-one correspondence [3] with linear mappings $\Lambda\colon \bar{\mathfrak{g}} \to \mathfrak{gl}(\mathfrak{m})$ such that $\Lambda|_{\mathfrak{g}} = \lambda$ and Λ is $\mathfrak{g}$-invariant. We call this mappings *(invariant) affine connections* on the pair $(\bar{\mathfrak{g}}, \mathfrak{g})$. The curvature and torsion tensors of the invariant affine connection Λ are given by the following formulas: $R\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{gl}(\mathfrak{m}),\ (x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto [\Lambda(x_1), \Lambda(x_2)] - \Lambda([x_1, x_2])$; $T\colon \mathfrak{m} \wedge \mathfrak{m} \to \mathfrak{m},\ (x_1+\mathfrak{g}) \wedge (x_2+\mathfrak{g}) \mapsto \Lambda(x_1)(x_2+\mathfrak{g}) - \Lambda(x_2)(x_1+\mathfrak{g}) - [x_1, x_2]_{\mathfrak{m}}$.

We restate the theorem of Wang on the holonomy algebra of an invariant connection: the Lie algebra of the holonomy group of the invariant connection defined by $\Lambda : \bar{\mathfrak{g}} \to \mathfrak{gl}(3, \mathbb{R})$ on $(\bar{\mathfrak{g}}, \mathfrak{g})$ is given by $V + [\Lambda(\bar{\mathfrak{g}}), V] + [\Lambda(\bar{\mathfrak{g}}), [\Lambda(\bar{\mathfrak{g}}), V]] + \ldots$, where V is the subspace spanned by $\{[\Lambda(x), \Lambda(y)] - \Lambda([x, y]) | x, y \in \bar{\mathfrak{g}}\}$.

We describe all local three-dimensional homogeneous spaces, allowing affine connections, it is equivalent to the description of effective pairs of Lie algebras, and all invariant affine connections on the spaces together with their curvature, torsion tensors and holonomy algebras. We use the algebraic approach for description of connections, methods of the theory of Lie groups, Lie algebras and homogeneous spaces.

References

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