Isomorphism problem for Cayley combinatorial objects

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,
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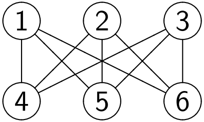
Graph Isomorphism.

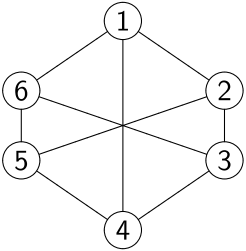
$\Gamma_1 = (\Omega_1, E_1) \cong \Gamma_2 = (\Omega_2, E_2)$ iff there exists a bijection (an isomorphism) $f : \Omega_1 \to \Omega_2$ such that

$$\forall \alpha_1, \beta_1 \in \Omega_1 : \quad (\alpha_1^f, \beta_1^f) \in E_2 \quad \Leftrightarrow \quad (\alpha_1, \beta_1) \in E_1.$$

$\mathrm{Aut}(\Gamma_1)$ is the automorphism group of Γ_1.

Example





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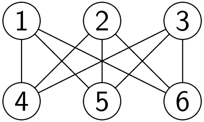

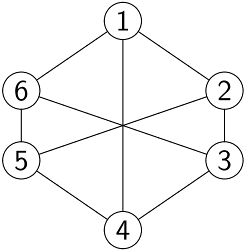

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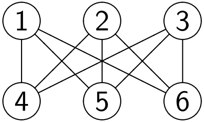

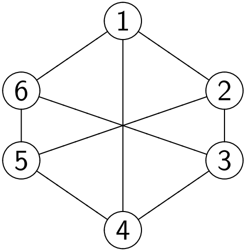

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$$\mathrm{Aut}(\Gamma) = (S_3 \times S_3).S_2.$$

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Theorem (L.Babai, 2015).

The isomorphism of n-vertex graphs can be tested in time $\exp(O((\log n)^c))$.

Cayley graphs

Definition

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Any combinatorial object (H, O) is called a Cayley object if it is H_R-invariant.

Theorem

Subidussi's Theorem is true for any combinatorial object.

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Given two Cayley graphs Cay(H, S) and Cay(H, T). Find whether they are isomorphic.

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Cayley graph recognition problem

Given a graph Γ of order n and a finite group $H, |H| = n$. Find whether Γ is isomorphic to a Cayley graph over H.

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Example

Take $H = \mathbb{Z}_2 \times \mathbb{Z}_4$, $S = H \setminus K$, $T = H \setminus C$ where K, C are order 4 subgroups, $K \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $C \cong \mathbb{Z}_4$. Then

$$\mathrm{Cay}(H,S) \cong K_{4,4} \cong \mathrm{Cay}(H,T).$$

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CIG-groups

A group H is called a CI-group with respect to graphs, CIG-group for short, if any subset of H is a CI-subset.

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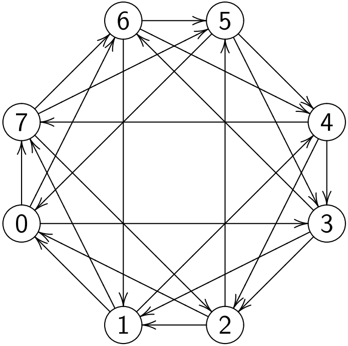

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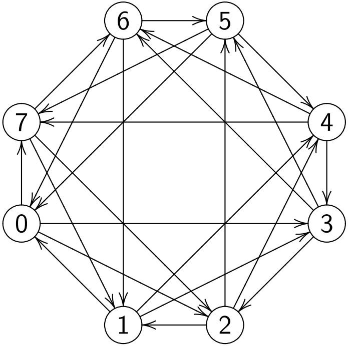

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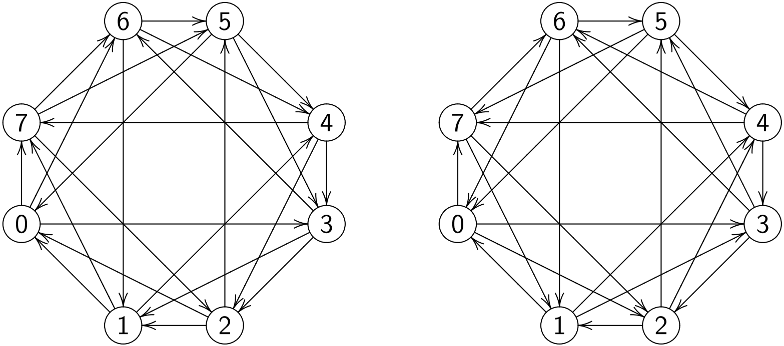

$\mathrm{Cay}(\mathbb{Z}_8, \{1, 2, 5\}) \xrightarrow{(2,6)(3,7)} \mathrm{Cay}(\mathbb{Z}_8, \{1, 6, 5\})$

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1. n is a prime - Elspas & Parsons;
2. $n = 2p, 3p, 4p$ - Babai 1977;
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Pálfy's correction of Ádám's conjecture (1987):

Ádám conjecture is true if n is a square free or twice square free number.

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Theorem (Muzychuk, 1995-97)

The corrected Ádám's conjecture is true.

CI-groups w.r.t. digraphs

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During last 40 years the classification of CIG-groups was studied by many researches: B. Alspach, L.Babai, M.Conder, E. Dobson, B. Elspas, V.N.Egorov, P.Frankl, C. Godsil, M. Hirasaka, M. Klin, I. Kovacs, C.H.Li, Z.P. Lu, A.I.Markov, L. Nowitz, T.D. Parsons, P. Pálfy, R. Pöschel, C. Praeger, P. Spiga, G. Somlai, J. Turner.

Theorem (necessary conditions to be a CIG-group)

If H is a CI-group w.r.t. digraphs, then H is a coprime product of groups from the following list:

$$\mathbb{Z}_p^e, \quad \mathbb{Z}_4, Q_8, A_4, E(M,2), E(M,3), E(M,4).$$

where M is a direct product of elementary abelian groups of odd order.

CI-groups w.r.t. digraphs (sufficient conditions)

Theorem

The following groups are CI-groups w.r.t. digraphs

1. $\mathbb{Z}_n$ where n is square-free or twice square-free number;
2. $\mathbb{Z}_p^e, e \leq 4$ and $\mathbb{Z}_2^5$;
3. $\mathbb{Z}_p^2 \times \mathbb{Z}_q, \mathbb{Z}_p^3 \times \mathbb{Z}_q$ where p and q are distinct primes;
4. $D_{2p}, \mathbb{F}_{3p}, \mathbb{Z}_p \rtimes \mathbb{Z}_4$;
5. Q_8, A_4.

1. C.H. Li, On isomorphisms of finite Cayley graphs - survey, DM 256 (2002),
2. C.H. Li, Z.P. Lu, P. Pálfy, Further restrictions on the struture of finite CI-groups, JACO 26 (2007).

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6. Find a polynomial algorithm which solves a recognition problem for Cayley graphs over $\mathbb{Z}_p^2$.

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3. $n = 2^m$, Muzychuk & Pöschel, 1999;

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2. $n = p^m, p > 2$, Klin & Pöschel, 1978;
3. $n = 2^m$, Muzychuk & Pöschel, 1999;
4. arbitrary n, Evdokimov & Ponomarenko - 2003, Muzychuk, 2004.

An isomorphism problem for arbitrary cyclic combinatorial objects of orders p^2 and pq was solved by Job, Huffman and Pless in 1993,1996.

Solving sets

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$$\iff \exists_{p \in P} \mathrm{Cay}(H, S)^p = \mathrm{Cay}(H, T).$$

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A group H is a CIG-group iff $\mathrm{Aut}(H)$ is a solving set for H.

"Individual" solving set

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Theorem (Babai, 1977)

A subset $S \subset H$ is a CI-subset iff any regular subgroup of $\mathrm{Aut}(\mathrm{Cay}(H, S))$ isomorphic to H is conjugate to H_R in $\mathrm{Aut}(\mathrm{Cay}(H, S))$.

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Definition

Let $G \leq \mathrm{Sym}(H)$ be an arbitrary group. A set $F_i, i \in I$ of H-regular subgroups of G is called an H-base of G iff any H-regular subgroup of G is conjugate in G to exactly one F_i.

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Theorem

Let S be an arbitrary subset of H. Let $F_i, i \in I$ be an H-base of the group $G := \mathrm{Aut}(\mathrm{Cay}(H,S)))$. Denote by $f_i \in \mathrm{Sym}(H)$ permutations such that $H_R = F_i^{f_i}, i \in I$. Then $\cup_{i \in I} f_i \mathrm{Aut}(H)$ is a solving set for $\mathrm{Cay}(H,S)$.

Example

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5. $\langle\rho\rangle = \langle\sigma\rangle^{(2,6)(3,7)} \implies \mathrm{Aut}(\mathbb{Z}_8) \cup (2, 6)(3, 7)\,\mathrm{Aut}(\mathbb{Z}_8)$ is a solving set for $\mathrm{Cay}(\mathbb{Z}_8, \{1, 2, 5\})$.

How to construct a solving set for all Cayley graphs over H

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4. Klin-Pöschel approach - use the method of Schur rings to find all possible automorphism groups.

Automorphism groups of Cayley graphs

Proposition

Let $G = \text{Aut}(\text{Cay}(H, S))$ and $\mathcal{S} = \{S_0, S_1, ..., S_d\}$ be the set of orbits of the point stabilizer G_e. Then

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Theorem, I. Schur (1933)

The partition $\mathcal{S}$ gives rise to a subalgebra of a group algebra $\mathbb{Q}[H]$.

These special partitions will be called Schur partitions of H.

Schur rings (algebras)

Definition (Wielandt)

Let $S \subseteq H$. An element $\underline{S} := \sum_{s \in S} s \in \mathbb{Q}[H]$ is called a simple quantity. We abbreviate $\underline{\{g\}}$ as g .

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A subalgebra $\mathcal{A}$ of $\mathbb{Q}[H]$ arising in this way is called a Schur algebra/ring.

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   3. Set $P(\mathcal{S}) := \bigcup_{i=1}^{k} f_i \,\mathrm{Aut}(H)$;
3. Take $\bigcup_{\mathcal{S} \in \mathfrak{S}} P(\mathcal{S})$ as a solving set for Cayley digraphs over H.

This scheme successfully worked for $\mathbb{Z}_n$ if n is a power of an odd prime or a product of two distinct primes.

Example: solving set for circulant graphs of order 8

The following list was generated by the computer program COCO (thanks to Misha Klin).

$\{0\}$, $\{1, 2, 3, 4, 5, 6, 7\}$;
$\{0\}$, $\{1, 3, 5, 7\}$, $\{2, 6, 4\}$;
$\{0\}$, $\{1, 3, 5, 7, 2, 6\}$, $\{4\}$;
$\{0\}$, $\{1, 3, 5, 7\}$, $\{2, 6\}$, $\{4\}$;
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Example

N	S-partition $\mathcal{S}$	Aut. group	cyclic bases	Solving set
1	$\{0\}, \{1,2,3,4,5,6,7\}$	S_8	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
2	$\{0\}, \{1,3,5,7\}, \{2,6,4\}$	$S_2 \wr S_4$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
3	$\{0\}, \{1,3,5,7,2,6\}, \{4\}$	$S_4 \wr S_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
4	$\{0\}, \{1,3,5,7\}, \{2,6\}, \{4\}$	$S_2 \wr S_2 \wr S_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
5	$\{0\}, \{1,3,5,7\}, \{2\}, \{6\}, \{4\}$	$S_2 \wr \mathbb{Z}_4$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
6	$\{0\}, \{1,5\}, \{3,7\}, \{2\}, \{6\}, \{4\}$	$\mathbb{Z}_8.Z_2$	$\langle\rho\rangle, \langle\sigma\rangle$	$\mathbb{Z}_8^* \cup \alpha\mathbb{Z}_8^*$
7	$\{0\}, \{1,5\}, \{3,7\}, \{2,6\}, \{4\}$	$\mathbb{Z}_4 \wr S_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
8	$\{0\}, \{1,3\}, \{5,7\}, \{2,6\}, \{4\}$	$\mathbb{Z}_8.\mathbb{Z}_2$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
9	$\{0\}, \{1,7\}, \{3,5\}, \{2,6\}, \{4\}$	D_{16}	$\langle\rho\rangle$	$\mathbb{Z}_8^*$
10	$\{0\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}$	$\mathbb{Z}_8$	$\langle\rho\rangle$	$\mathbb{Z}_8^*$

Here $\alpha = (2,6)(3,7)$. Thus $\mathbb{Z}_8^* \cup \alpha\mathbb{Z}_8^*$ is a solving set for circulant graphs over $\mathbb{Z}_8$.

Solution of the isomorphism problem for circulant digraphs.

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Theorem (Klin-Pöschel, 1978)

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1. the number of Schur rings over $\mathbb{Z}_n$ is bounded by $n^C, 2 \leq C < 2.5$;

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Theorem (Muzychuk-Pöschel, 1999)

Let $n = 2^m$. Then there exists an efficiently constructed solving set P_n for circulant graphs of order n s.t. $|P_n| \leq n^C \varphi(n)$.

But if n is a square-free number, the number of Schur partitions growes exponentially!

Control of H-bases

Definition

Let $H_R \leq X \leq Y \leq \operatorname{Sym} H$ be arbitrary subgroups. We say that X controls H-bases of Y, notation $X \preceq_H Y$, if there exists an H-base of X which contains an H-base of Y.

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Proposition

The following are equivalent

1. $X \preceq_H Y$;
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Proposition

The relation $\preceq_H$ is a partial order on the lattice $[H_R, \operatorname{Sym}(H)]$.

$\preceq_H$-minimal subgroups.

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Example. The symmetric group $\operatorname{Sym}(8)$ has two $\prec_{\mathbb{Z}_8}$-minimal subgroups: $\mathbb{Z}_8$ and $\mathbb{Z}_8 \rtimes \langle \sigma \rangle$ where $\sigma(x) = 5x$.

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Theorem (Palfy, 1987)

If H is a cyclic group of order n, then H_R is a unique $\preceq_H$-minimal subgroup iff $n = 4$ or $\gcd(n, \varphi(n)) = 1$.

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Theorem (M., 1999)

If H is cyclic, then each $\prec_H$-minimal subgroup of $X \in [H_R, \mathrm{Sym}\, H]$ is solvable and $\pi(X) = \pi(H)$.

Isomorphism problem for cyclic combinatorial objects

Theorem (M., 2004)

The automorphism group G of a colored circulant digraph contains a nilpotent subgroup which controls cyclic bases.

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Theorem (M. 2004)

Let $n = p_1^{m_1} \cdot \ldots \cdot p_k^{m_k}$ be a decomposition of n into a product of prime powers. Denote by $P_{p_i^{m_i}}$ a solving set for colored circulant digraphs over $\mathbb{Z}_{p_i^{m_i}}$.Then the set $P_n := P_{p_1^{m_1}} \times \ldots \times P_{p_k^{m_k}}$ is a solving set for colored Cayley digraphs over $\mathbb{Z}_n$. In particular, $|P_n| < n^C \varphi(n)$.

Non-graphical cyclic combinatorial objects

Theorem (M., 2011)

The set P_n is also a solving set for a semisimple cyclic codes of length n. In other words, two semisimple cyclic codes $C, D \leq \mathbb{F}_q^n$ are permutation equivalent iff there exists $g \in P_n$ s.t. $C^g = D$.

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Theorem (I. Kovacs, D. Marušič and M. Muzychuk, 2015)

A cyclic group is a CI-group with respect to balanced configurations.

Non-graphical combinatorial objects: Cayley maps

Definition

A Cayley map is a triple $M(H, S, \rho)$ where

1. H is a finite group;
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Example: $H = \mathbb{Z}_2 \times \mathbb{Z}_2, S = \{01, 10, 11\}, \rho = (01, 10, 11)$.

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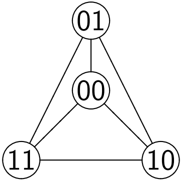

Map isomorphisms

Definition

Two Cayley maps $M(H, S, \rho)$ and $M(H, S', \rho')$ are isomorphic iff there exists a bijection $f \in \mathrm{Sym}(H)$ s.t.

$$\{(h, sh, \rho(s)h) \mid s \in S, h \in H\}^f = \{(h, sh, \rho'(s)h) \mid s \in S', h \in H\}.$$

Cayley isomorphism

Two Cayley maps $M(H, S, \rho)$ and $M(H, S', \rho')$ are Cayley isomorphic iff there exists $f \in \mathrm{Aut}(H)$ s.t. $S^f = S$ and $f_S\rho = \rho' f_S$.

Problem

Classify all finite groups with CI-property with respect to maps.

CI-groups with respect to maps

Theorem (M and G. Somlai, 2015)

Let H be a CI-group with respect to Cayley maps. Then H is isomorphic to one of the following groups

1. $\mathbb{Z}_2^r \times \mathbb{Z}_m, \mathbb{Z}_4 \times \mathbb{Z}_m, \mathbb{Z}_8 \times \mathbb{Z}_m, Q_8 \times \mathbb{Z}_m$;
2. $\mathbb{Z}_m \rtimes \mathbb{Z}_{2^e}, e = 1, 2, 3$.

where m is a square-free odd number.

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