Hamilton Cycles in Graphs Embedded into Surfaces

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Although hundreds of papers deals with the problem of existence of a Hamiltonian cycle in a graph, there is a luck of results on the hamiltonicity of cubic graphs. Among others, it is well-known that to decide whether a cubic graph is Hamiltonian is an NP-complete problem. The main idea of the talk is to present a new approach to investigate hamiltonicity of graphs. Instead of graphs, we consider graphs embedded into closed surfaces such that each face is bounded by a circuit (no repetitions of vertices are allowed). Such an embedding is called polytopal or circular. By the cycle-double-cover conjecture every 2-connected graph admits a polytopal embedding. In an embedded graph the set of hamilton cycles split into three classes: contractible, bounding but non-contractible cycles and non-separating cycles. We shall investigate bounding and contractible hamilton cycles.

Assume first that the underlying surface is sphere. Due to the Jordan curve theorem a Hamilton cycle C in a spherical map $\mathcal{M}$ separates the surface into two disks bounded by C. Consider the two disjoint sets of faces $\mathcal{A}$ and $\mathcal{B}$ separated by C. Then it is not difficult to see that the corresponding sets of vertices $\mathcal{A}^*$ and $\mathcal{B}^*$ in the dual map induce two disjoint trees in $\mathcal{M}^*$. The main idea consists in reversing the above process. We shall try to identify a proper tree $T \subseteq \mathcal{M}$ (or a "one-face embedded" subgraph) in the dual $\mathcal{M}^*$, such that the topological closure of the faces in $\mathcal{M}$ corresponding to the vertices of T will form a bordered surface with a connected boundary creating a bounding hamilton cycle in $\mathcal{M}$. We shall call the tree T a co-hamiltonian tree. Now, let $\mathcal{M}$ be a polytopal map. In general, a bounding hamilton cycle in $\mathcal{M}$ will always define a bi-partition of the vertex set of the dual map. In order to understand what sort of bi-partitions give a hamilton cycle in the original map, we introduce a useful concept of a 1-sided subgraph which generalizes the concept of an embedded tree. Using this concept we were able to prove a theorem stating that a map $\mathcal{M}$ on a surface S admits a bounding hamilton cycle if and only if the vertex set of the dual $\mathcal{M}^*$ admits a partition into two subsets which induce one-sided subgraphs H and K such that $\beta(H) + \beta(K) = \epsilon(S)$, where $\beta(H)$ and $\beta(K)$ are the Betti numbers and $\epsilon(S)$ is the Euler genus. The hamilton cycle is contractible if and only if one of the subgraphs H, K is a tree. The two subgraphs satisfying the statement will be called co-hamiltonian subgraphs.

Under certain circumstances we can guarantee existence of a vertex-bipartition in the dual map into two co-hamiltonian subgraphs, or we can prove that such a decomposition cannot exist. For instance, we show that the truncation of a triangulation without a separating 3-cycle has a hamilton path and if the number of triangles is congruent 2 mod 4 it has a bounding hamilton cycle. Also we shall deal with truncations of triangulations with faces of size at most 7. We show that such map is either hamiltonian, if the number of triangles is congruent 2 mod 4, or it has a hamilton path. This relates the result to a conjecture by Barnette, recently proved by Kardoš, stating that cubic polyhedral graphs with faces of size at most six are hamiltonian. Also we present a uniform approach to the problem of hamiltonicity of Cayley graphs coming from groups of the form

$$\langle x, y \mid y^2 = (xy)^3 = 1, \dots \rangle,$$

investigated in papers by Glower, Youngs, Marušič, Kutnar and Malnič. A new result proves hamiltonicity, or at least existence of a hamilton path in Cayley graphs generated by three involutions x, y and z satisfying the relations $(xy)^3 = (yz)^3 = 1$. These are particular instances of a folklore conjecture stating that Cayley graphs are hamiltonian which solution does not seem to be in hand.

References

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