Φ-Harmonic Functions on Graphs

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Let Φ be a function with some special properties. Properly speaking, it is an N-function. In our talk we will consider a number of aspects of Φ-harmonic analysis on graphs. In particular, we will introduce the key definitions and will reveal that the ones in question are well-defined. Also we will give an overview of our results that bring discrete analogs of classical theorems for harmonic function in the usual sense: uniqueness theorem, Harnack's inequality, Harnack's principle. Our work generalizes results obtained in:

Holopainen, Ilkka, and Soardi, Paolo M.. "p-harmonic functions on graphs and manifolds Manuscripta mathematica 94.1 (1997): 95–110.

N-functions

Definition: N-function

A function $\Phi : \mathbb{R} \to \mathbb{R}$ is said to be *N-function* if it admit the following representation

$$\Phi(x) = \int\limits_0^{|x|} \varphi(t)\, dt,$$

where $\varphi(t)$ is defined for $t \geqslant 0$, non-decreasing, left continuous, satisfying the properties $\varphi(t) > 0$ as $t > 0$; $\varphi(0) = 0$; $\lim_{t\to\infty} \varphi(t) = \infty$. From now on, we will write Φ' instead of φ. Therefore, for N-function Φ the following holds:

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- $\Phi(x) > 0$, if $x > 0$;
- Φ is even and convex;;
- $\lim\limits_{x\to 0} \frac{\Phi(x)}{x} = 0$, $\lim\limits_{x\to\infty} \frac{\Phi(x)}{x} = +\infty$.

N-functions

Definition: Complimentary N-function

Let Φ be an N-function, the function given by

$$\Psi(x) = \int\limits_0^x (\Phi')^{-1}(t)\, dt, \quad \text{where } (\Phi')^{-1}(x) = \sup_{\Phi'(t)\le x} t,$$

is called *complementary for* Φ.

Φ-Harmonic Functions

Let $\Gamma = (V,\ E)$ be connected infinite graph of bounded degree (with no self-loops), where V is the vertex set, and E is the edge set. The notation $x \sim y$ stands for a couple $(x,\ y)$ of adjacent vertices, $(x,\ y) = e \in E$.

Now given a function $f\colon S \cup \partial S \to \mathbb{R}$, where $S \subset V$ and $\partial S = \bigcup_{x \in S}\{y \in V \backslash S \mid y \sim x\}$, we introduce a list of definitions

The classical definition of harmonic function $f(x)$ on graph requires that the equation

$$f(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$

holds at every x. It is clear that the mentioned condition just means

$$\sum_{y \sim x} (f(y) - f(x)) = 0$$

This one is called the discreet laplacian

$$\Delta f(x) = \sum_{x \sim y} (f(y) - f(x))$$

Φ-Harmonic Functions

Definition: Φ-Laplacian

The operator $\mathbb{R}^{S\cup\partial S} \xrightarrow{\Delta_\Phi} \mathbb{R}^{S\cup\partial S}$, defined by

$$\Delta_\Phi f(x) = \sum_{x\sim y} \Phi'(f(y) - f(x))$$

is called $\Phi-$laplacian.

Definition: Φ-Harmonic functions

A function f is said to be $\Phi-$harmonic in S, if $\Delta_\Phi f(x) = 0$ holds far all $x \in S$. We denote by $\mathcal{H}^\Phi(S)$ the set of all such functions.

Φ-Harmonic Functions

Introduce the functional $\mathbb{R}^{S \cup \partial S} \xrightarrow{\rho} \mathbb{R}^{\geq 0}$ as the following equation

$$\rho(f) = \sum_{x \in S} \sum_{y \sim x} \Phi(f(y) - f(x))$$

Below we will use the notation

$$\langle f, g \rangle(x, y) = \Phi'(f(y) - f(x))(g(y) - g(x)).$$

Given a couple of function defined in S, put

$$\langle \Delta_\Phi h, f \rangle = \sum_{x \in S} \sum_{y \sim x} \langle h, f \rangle(x, y)$$

Φ-Harmonic Functions

Definition: Weak harmonicity

We say that a function h is weakly Φ-harmonic if $\langle \Delta_\Phi h,\ f \rangle = 0$ for all $f \colon S \cup \partial S \to \mathbb{R}$ such that $f|_{\partial S} = 0$.

The following lemma reveals relations between two definitions of Φ-harmonicity above.

Lemma 1

Let $S \subset V$ be a finite set. Then property to be Φ-harmonic in a weak sense is equivalent to Φ-harmonicity. Put it otherwise, $\Delta_\Phi f = 0$ if and only if $\langle \Delta_\Phi f,\ g \rangle = 0$ for all $g \colon S \cup \partial S \to \mathbb{R}$ such that $g|_{\partial S} = 0$.

Φ-Harmonic Functions

Now we can clarify the role played by the functional ρ mentioned above.

Lemma 2

Suppose $S \subset V$ is a finite set. The equation $\Delta_\Phi f = 0$ holds if and only if f minimizes $\rho(g)$ over the set $M(f) = \{g \colon S \cup \partial S \to \mathbb{R} \mid g|_{\partial S} = f|_{\partial S}\}$

Φ-Harmonic Functions

Suppose S is a finite set. Let $\{f_i\}$ be a sequence of functions in $S \cup \partial S$, which converges pointwise to a function f, then it is not hard to see

$$\rho(f_i) \to \rho(f),\ \Delta_\Phi f_i(x) \to \Delta_\Phi f(x)$$

Theorem 1

Let S be finite. Given an arbitrary function f in ∂S, there is a unique function h in $S \cup \partial S$ such that h is Φ-harmonic in S and $h|_{\partial S} = f$.

Φ-Harmonic Functions

Definition: Super(Sub)harmonisity

We say that h is Φ-superharmonic (subharmonic) in U if $\Delta_\Phi h(x) \leq 0$ (resp. $\Delta_\Phi h(x) \leq 0$) at every point $x \in U$.
It is not hard to show that this condition is equivalent to

$$\langle \Delta_\Phi h,\, f \rangle \geq 0 \text{ (resp. } \leq 0)$$

for all $f\colon U \cup \partial U \to \mathbb{R}^+$ such that $f|_{\partial U} = 0$ and f has finite support .

Φ-Harmonic Functions

Theorem 2

Let f be Φ-superharmonic and g be Φ-subharmonic in a finite set S such that $f \geq g$ in ∂S. Then $f \geq g$ in S.

Corollary

Suppose f and g are Φ-harmonic functions in a finite set S such that $f|_{\partial S} = g|_{\partial S}$. Then $f = g$ in S.

Φ-Harmonic Functions

Henceforth $U \subset V$ is an arbitrary set needed not be finite.

Theorem 3: Harnack's inequality

Let Φ and Ψ be a couple of complementary $N-$functions, $h \colon U \cup \partial U \to \mathbb{R}^+$ is Φ-superharmonic in U. Then the following estimation holds at every point $x \in U$

$$\max_{y \sim x} h(y) \leq [\Psi'(\deg(x)) + 1] h(x)$$

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Lemma 3

Let $\{S_i\}$ be an increasing sequence of finite connected subset of V, and let $U = \bigcup_i S_i$. Suppose $\{h_i\}$ is a sequence of functions in $U \cup \partial U$ such that $h_i(x) \to h(x) < \infty$ for all $x \in U \cup \partial U$. If h_i is Φ-harmonic (resp. Φ-superharmonic, Φ-subharmonic) in every S_i, then h is Φ-harmonic (resp. Φ-superharmonic, Φ-subharmonic) in U.

Theorem 4: Harnack's principle

Let S_i and U be as above . Let $\{h_i\}$ be an increasing sequence of functions in $U \cup \partial U$. If h_i is Φ-harmonic (or Φ-superharmonic) in every S_i, then either $h_i(x) \to \infty$ for every $x \in U$, or $h_i(x) \to h(x)$ for all $x \in U$ and h— Φ-harmonic (resp. Φ-superharmonic) in U.

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Thank you for your attention!