About group density function

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Let G be a group and $\mathfrak{N} = \{g_1, ..., g_n\}$ be a set of generators of G. Following [1, p. 102] define

$$F_{(G,\mathfrak{N})}(l) = \begin{cases} 1, \text{ if } l = 0, \\ \text{the number of elements of } G \text{ whose irreducible length in at most l}, \text{ if } l > 0 \end{cases}$$

to be a function of group growth of G on the set $\mathfrak{N}$. Let

$$P_{(G,\mathfrak{N})}(l) = \begin{cases} 1, \text{ if } l = 0, \\ F_{(G,\mathfrak{N})}(l) - F_{(G,\mathfrak{N})}(l-1), \text{ if } l > 0 \end{cases}$$

be a *group density function* of G on the set of generators $\mathfrak{N}$.

Question. Let G and H be groups, $\mathfrak{N}$ and $\mathfrak{M}$ be sets of generators of G and H respectively. Suppose, $P_{(G,\mathfrak{N})}(l) = P_{(H,\mathfrak{M})}(l)$. Are groups G and H isomorphic?

We prove the following theorems.

Theorem 1. *Let G and H be groups, $\mathfrak{N}$ and $\mathfrak{M}$ be sets of generators of G and H respectively. If $P_{(G,\mathfrak{N})}(l) = P_{(H,\mathfrak{M})}(l)$ then $|\mathfrak{N}| = |\mathfrak{M}|$.*

Theorem 2. *Let G and H be groups, $\mathfrak{N}$ and $\mathfrak{M}$ be sets of generators of G and H respectively. If $P_{(G,\mathfrak{N})}(l) = P_{(H,\mathfrak{M})}(l)$ then $|G| = |H|$.*

A set $\mathfrak{N} = \{g_1, ..., g_n\}$ of generators of a group G is called *independent* if $\langle \mathfrak{N} \setminus \{g_i\} \rangle \cap \langle g_i \rangle = \langle e \rangle$ is a trivial subgroup for all $i \in \{1, ..., n\}$.

We prove the following theorem.

Theorem 3. *Let G and H be finite abelian p-groups, $\mathfrak{N}$ and $\mathfrak{M}$ be independent sets of generators of G and H respectively. If $P_{(G,\mathfrak{N})}(l) = P_{(H,\mathfrak{M})}(l)$ then $G \cong H$.*

The most interesting case of the Question is the case when G and H are finite simple non-abelian groups, $\mathfrak{N}$ and $\mathfrak{M}$ are independent sets of their generators.

Conjecture. *Let G and H be finite non-abelian simple groups, $\mathfrak{N}$ and $\mathfrak{M}$ be independent sets of generators of G and H respectively. If $P_{(G,\mathfrak{N})}(l) = P_{(H,\mathfrak{M})}(l)$ then $G \cong H$.*

Theorem 4. *Let $A_8 = \langle \mathfrak{N} \rangle$ and $L_3(4) = \langle \mathfrak{M} \rangle$, where $\mathfrak{N}$ and $\mathfrak{M}$ are independent sets of generators. Then $P_{(A_8,\mathfrak{N})}(l) \neq P_{(L_3(4),\mathfrak{M})}(l)$.*

References

[1] O. V. Melnikov, V. N. Remeslennikov, V. A. Romankov, L. A. Skornyakov, I. P. Shestakov, General Algebra. *Science, Moscow* (1990) 552.