Small cycles in the Bubble-Sort graph

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We investigate the Bubble-Sort graph BS_n, $n \geqslant 2$, that is the Cayley graph on the symmetric group Sym_n generated by transpositions from the set $t = \{t_{ii+1} \in Sym_n, 1 \leqslant i \leqslant n-1\}$. In 2006 Yosuke Kikuchia and Toru Arakib have shown [1] that BS_n, $n \geqslant 4$, contains all cycles of even length l, where $4 \leqslant l \leqslant n!$. However a characterization of these cycles has not been done.

In this talk, a characterization of small cycles is given by their canonical forms. A sequence of transpositions $C_l = t_{i_0 i_0+1} ... t_{i_{l-1} i_{l-1}+1}$, where $1 \leqslant i_j \leqslant n-1$, and $i_j \neq i_{j+1}$ for any $j \in \{0, ..., l-1\}$, such that $\pi t_{i_0 i_0+1} ... t_{i_{l-1} i_{l-1}+1} = \pi$, where $\pi \in Sym_n$, is said to be a *form of a cycle* C_l of length l in the Bubble-Sort graph. *The canonical form* C_l of an l-cycle is called a form with a lexicographically maximal sequence of indices. For cycles of a form $C_l = t_{ii+1} t_{jj+1} ... t_{ii+1} t_{jj+1}$, where $l = 2k$, and $t_{ii+1} t_{jj+1}$ appears k times, we write $C_l = (t_{ii+1} t_{jj+1})^k$.

The following results are obtained.

Theorem 1. *Each of the vertices of the Bubble-Sort graph BS_n, $n \geqslant 4$, belongs to $\frac{(n-2)(n-3)}{2}$ different 4-cycles of the canonical form $C_4 = (t_{ii+1} t_{jj+1})^2$, where $1 \leqslant i < j-1 \leqslant n-1$. Totally, there are $\frac{(n-2)(n-3)n!}{8}$ different cycles of length four in the graph.*

Theorem 2. *Each of the vertices of the Bubble-Sort graph BS_n belongs to $(n-2)$ 6-cycles of the canonical form*

$$C_6^1 = (t_{i+1i+2} t_{ii+1})^3, \;\; 1 \leqslant i \leqslant n-2, \;\; n \geqslant 3;$$

to $(n-4)(n-3)$ 6-cycles of the canonical form

$$C_6^2 = (t_{ii+1} t_{i+1i+2})(t_{jj+1})(t_{i+1i+2} t_{ii+1})(t_{jj+1}), 1 \leqslant i < j \leqslant n-1, n \geqslant 5,$$

and to $\frac{(n-3)(n-4)(n-5)}{6}$ 6-cycles of the canonical forms

$$C_6^3 = (t_{ii+1} t_{jj+1} t_{kk+1})^2, k-1 > j > i+1, n \geqslant 6;$$

$$C_6^4 = (t_{ii+1} t_{jj+1} t_{ii+1} t_{kk+1} t_{jj+1} t_{kk+1}), k-1 > j > i+1, n \geqslant 6.$$

Totally, there are $\frac{(n^3-9n^2+29n-30)n!}{18}$ cycles of length six in the graph.

Analogous results about cycles of length eight are presented by **Theorem 3** in the talk.

References

[1] Yosuke Kikuchia, Toru Arakib, Edge-bipancyclicity and edge-fault-tolerant bipancyclicity of bubble-sort graphs. *Information Processing Letters* **2** (2006) 52–59.