Small cycles in the Bubble-Sort graph

Simonova Anna

Graphs and Groups, Spectra and Symmetries

NSU, 2016

We investigate the Bubble-Sort graph BS_n, $n \geqslant 2$, that is the Cayley graph on the symmetric group Sym_n generated by transpositions from the set $t = \{t_{ii+1} \in Sym_n, 1 \leqslant i \leqslant n-1\}$. In 2006 Yosuke Kikuchia and Toru Arakib have shown [1] that BS_n, $n \geqslant 4$, contains all cycles of even length l, where $4 \leqslant l \leqslant n!$.

However a characterization of these cycles has not been done.

---

[1] Yosuke Kikuchia, Toru Arakib. Edge-bipancyclicity and edge-fault-tolerant bipancyclicity of bubble-sort graphs // Information Processing Letters Volume 100, Issue 2, 31 October 2006, Pages 52–59.

Definitions

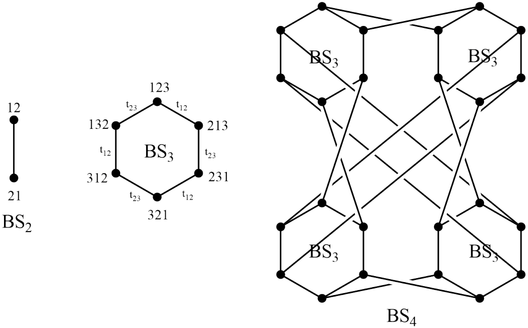
The Symmetric group Sym_n is the group of all permutations acting on the set {1,..,n}. $|Sym_n| = n!$

The Bubble-Sort graph $BS_n = Cay(Sym_n, t), n \geqslant 2$, on the symmetric group Sym_n with the generating set

$$t = \{t_{ii+1} \in Sym_n, 1 \leqslant i \leqslant n-1\},$$

where t_{ii+1} are 2-cycles interchanging ith and $(i+1)$th elements of a permutation when multiplied on the right.

BS_n graph is vertex-transitive, connected, bipartite $(n-1)$-regular graph with diameter $\frac{n(n-1)}{2}$; The graph has $n!$ vertices and $\frac{n(n-1)!}{2}$ edges.



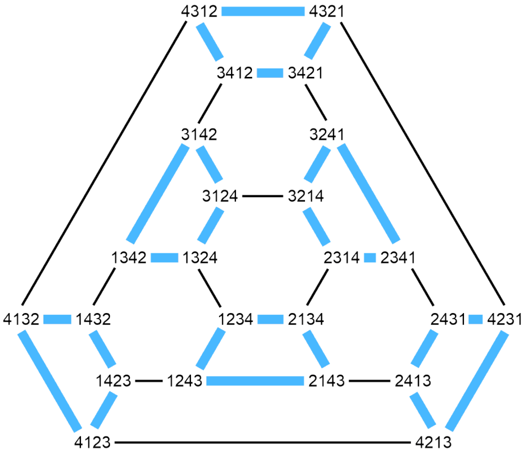
Examples of Bubble-sort graphs BS_2, BS_3 and BS_4

Form of a cycle C_l

A sequence of transpositions $C_l = t_{i_0\, i_0+1} \ldots t_{i_{l-1}\, i_{l-1}+1}$, where $1 \leqslant i_j \leqslant n-1$, and $i_j \neq i_{j+1}$ for any $j \in \{0, \ldots, l-1\}$, such that $\pi t_{i_0\, i_0+1} \ldots t_{i_{l-1}\, i_{l-1}+1} = \pi$, where $\pi \in Sym_n$, is said to be a ***form of a cycle*** C_l of length l in the Bubble-Sort graph.

The canonical form C_l of an l-cycle is called a form with a lexicographically maximal sequence of indices. For cycles of a form $C_l = t_{ii+1} t_{jj+1} \ldots t_{ii+1} t_{jj+1}$, where $l = 2k$, and $t_{ii+1} t_{jj+1}$ appears k times, we write $C_l = (t_{ii+1} t_{jj+1})^k$.

Example: 4-cycles in BS_4 graph



For $C_4 = t_{34}t_{12}t_{34}t_{12}$ the canonical form will be $C_4 = (t_{34}t_{12})^2$

4-cycles characterization

Theorem 1. *Each of the vertices of the Bubble-Sort graph* BS_n, $n \geqslant 4$, *belongs to* $\frac{(n-2)(n-3)}{2}$ *different 4-cycles of the canonical form* $C_4 = (t_{ii+1} t_{jj+1})^2$, *where* $1 \leqslant i < j - 1 \leqslant n - 1$.

6-cycles characterization

Theorem 2. *Each of the vertices of the Bubble-Sort graph BS_n belongs to $(n-2)$ 6-cycles of the canonical form*

$$C_6^1 = (t_{i+1i+2}t_{ii+1})^3, \ \ 1 \leqslant i \leqslant n-2, \ \ n \geqslant 3; \tag{1}$$

to $(n-4)(n-3)$ 6-cycles of the canonical form

$$C_6^2 = (t_{ii+1}t_{i+1i+2})(t_{jj+1})(t_{i+1i+2}t_{ii+1})(t_{jj+1}), \ \ 1 \leqslant i < j \leqslant n-1, \ \ n \geqslant 5, \tag{2}$$

and to $\frac{(n-3)(n-4)(n-5)}{6}$ 6-cycles of the canonical forms

$$C_6^3 = (t_{ii+1}t_{jj+1}t_{kk+1})^2, \ \ k-1 > j > i+1, \ \ n \geqslant 6; \tag{3}$$

$$C_6^4 = (t_{ii+1}t_{jj+1}t_{ii+1}t_{kk+1}t_{jj+1}t_{kk+1}), \ \ k-1 > j > i+1, \ \ n \geqslant 6. \tag{4}$$

8-cycles characterization

Teorem 3. *Each of the vertices of the Bubble-Sort graph BS_n belongs to* $3(n-3)$ *8-cycles of the canonical form*

$$C_8^1 = (t_{i+2i+3}t_{ii+1}t_{i+2i+3})(t_{i+1i+2}t_{ii+1})^2(t_{i+1i+2}), \quad (1)$$

where $1 \leqslant i < n-3,\ n \geqslant 4;$

to $(n-3)(n-4)$ *8-cycles of the canonical form*

$$C_8^2 = (t_{i+1i+2}t_{ii+1})^2(t_{jj+1})(t_{i+1i+2}t_{ii+1})(t_{jj+1}), \quad (2)$$

where $j > i+2$, *or* $j < i-1,\ n \geqslant 5;$

to $(n-5)(n-4)$ *8-cycles of the canonical form*

$$C_8^3 = (t_{i+2i+3}t_{i+1i+2}t_{ii+1})(t_{jj+1})(t_{ii+1}t_{i+1i+2}t_{i+2i+3})(t_{jj+1}), \quad (3)$$

where $j > i+3$, *or* $j < i-1,\ n \geqslant 6,$

and $\frac{(n^4-22n^3+179n^2-662n+1032)}{24}$ *8-cycles of the canonical form*

$$C_8^4 = (t_{ii+1}t_{jj+1}t_{kk+1}t_{mm+1})(t_{\sigma(i)\sigma(i)+1}t_{\sigma(j)\sigma(j)+1}t_{\sigma(k)\sigma(k)+1}t_{mm+1}), \quad (4)$$

where $\sigma \in S_3$ *on the set* $\{i, j, k\},\ n \geqslant 8.$

The number of 4-, 6- and 8-cycles

Theorem 4. *BS_n graph, $n \geqslant 3$, contains:*

$\frac{(n-2)(n-3)n!}{8}$ *different cycles of length four,*

$\frac{n^3-9n^2+29n-30}{3}n!$ *cycles of length six,*

$\frac{n^4-22n^3+183n^2-678n+1044}{4}n!$ *cycles of length 8.*

The results obtained clearly demonstrate the complexity of a BS_n graph cyclic system. The method used for proving Theorem 1, Theorem 2 and Theorem 3 allows us to estimate and count the number of cycles of length ten, twelve or more passing through the same vertex.

Small cycles in the Bubble-Sort graph

Simonova Anna

Graphs and Groups, Spectra and Symmetries

NSU, 2016