Minimal generating systems and structure of Sylow 2-subgroups of alternating groups

$Syl_2 A_2^k$ and $Syl_2 A_n$

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The aim of this investigation is to research the structure of Sylow 2-subgroups of $A_n$ and to construct a minimal generating system for it. The authors of the article [1] didn’t prove minimality of the system of generators for the Sylow 2-subgroups of $A_n$, which was found by them, and it was not fully investigated in [1, 2]. The case of Sylow 2-subgroups is very special, because group $C_2 \wr C_2 \wr \ldots \wr C_2$ can acts by odd permutations on $X^k$ [2], where $X = \{0, 1\}$. Let us denote by $X[k]$ a regular binary rooted tree with $k$ levels.

There was a mistake in a statement about irreducibility of the generating system of $k + 1$ elements for $Syl_2 (A_2^k)$, which was in abstract [3] in year 2015.

**Theorem 1.** A maximal 2-subgroups $G_k$ of $Aut X[k]$ that acts by even permutations on $X^k$ have structure of inner semidirect product $G_k \cong (S_2 \wr S_2 \ldots \wr S_2) \times (C_2 \times \ldots \times C_2)$ and isomorphic to $Syl_2 A_2^k$.

In accordance with Legendre formula, the power of 2 in $2^k$ is $\frac{2^k-1}{2-1}$. We need to subtract 1 from $2^k - 1$, because we have only $\frac{2^k-1}{2}$ of permutations. $|G_k| = |S_2 \wr \ldots \wr S_2| \cdot |C_2 \times \ldots \times C_2| = |Syl_2 A_2^k| = 2^{2^k - 2}.$

**Statement 1.** A quotient group $G_k / G^2_k G'_k$ is isomorphic to $C_2 \times C_2 \times \ldots \times C_2$.

Proof follows from that $G^2_k G'_k \cong G^2_k \triangleleft G_k$ and $|G : G^2_k G'_k| = 2^k$. Also it was deduced that derived length of $Syl_2 A_2^k$ is not always equal to $k$, because in case $k = 2$ the Sylow subgroup of $A_2^k$ is $Syl_2 A_4 \cong K_4$ but $K_4$ is abelian group, so its derived length is 1. But it was said in Lemma 3 of [1] that derived length of $Syl_2 A_2^k$ is always equal to $k$ if $k \geq 1$, so it is a mistake.

**Main Theorem.** A minimal generating system of Sylow 2-subgroups of alternating group $A_2^k$ consists of $k$ elements.

There is Morse function [4] $f : D^2 \to \mathbb{R}$ which has $2^k$ points of maximum and a graph of Kronrod-Reeb [5], which is correspondent to it and is obtained by contraction of every set’s component of level $f^{-1}(c)$ in a point. The group of automorphism of this graph is isomorphic to $Syl_2 S_2^k$.

**References**


