On transversals in completely reducible quasigroups and in quasigroups of order 4

Anna Taranenko
Sobolev Institute of Mathematics SB RAS, Novosibirsk, Russia
taattg@mail.ru

An n-ary operation $f : \Sigma^n \to \Sigma$, where Σ is a set of cardinality q, is called an n-ary quasigroup of order q if in the equality $x_0 = f(x_1, \ldots, x_n)$ knowledge of any n elements of $x_0, x_1, \ldots, x_n$ uniquely specifies the remaining one.

A transversal in a n-ary quasigroup f of order q is a set of $(n+1)$-tuples $\left\{(a_0^i, a_1^i, \ldots, a_n^i)\right\}_{i=1}^{q}, a_k^i \in \Sigma$ such that $a_0^i = f(a_1^i, \ldots, a_n^i)$ for all $i \in \{1, \ldots, q\}$ and $a_k^i \neq a_k^j$ for all $i \neq j$ and $k \in \{0, \ldots, n\}$.

An n-ary quasigroup f is a composition of an $(n - m + 1)$- quasigroup h and an m-quasigroup g if there exists a permutation $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that for all $x_1, \ldots, x_n \in \Sigma$

$$f(x_1, \ldots, x_n) = h(g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(n)}).$$

An n-ary quasigroup f is called completely reducible if $n \leq 2$ or if it can be represented as a composition of $n - 1$ 2-ary quasigroups.

Although there exist completely reducible quasigroups without transversals, we prove that most of such quasigroups do have transversals.

Theorem 1. *Let f be a completely reducible n-ary quasigroup of order q. If n is odd then f has at least $(q \cdot q!)^{\frac{n-1}{2}}$ transversals. If n is even and the most external quasigroup in a decomposition of f has a transversal, then f has at least $(q \cdot q!)^{\left\lfloor \frac{n-1}{2} \right\rfloor}$ transversals.*

Also, using a result of [1] we prove the following theorem that sustains a conjecture about transversals in latin hypercubes proposed in [2].

Theorem 2. *If n is odd then every n-ary quasigroup of order 4 has a transversal.*

Acknowledgments. The work is supported by the Russian Science Foundation (grant 14–11–00555).

References

[1] D. S. Krotov, V. N. Potapov, n-Ary quasigroups of order 4. *SIAM J. Discrete Math.* **23(2)** (2009) 561–570.

[2] I. M. Wanless, Transversals in latin squares: a survey. *Surveys in Combinatorics 2011, London Math. Soc. Lecture Note Ser.,* **392** Cambridge: Cambridge University Press (2011) 403–437.