On transversals in completely reducible quasigroups and in quasigroups of order 4

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Let $\Sigma_q = \{0, \dots, q-1\}$. A binary quasigroup of order q is a binary operation $f : \Sigma_q^2 \to \Sigma_q$ such that for all $a, b \in \Sigma_q$ the equalities

$$f(x, a) = b, \quad f(a, y) = b$$

have unique solutions.

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0	1	2	3
1	0	**3**	2
2	3	0	**1**
3	**2**	1	0

0	1	2	3	**4**
1	0	3	4	2
2	**3**	4	0	1
3	4	1	**2**	0
4	2	**0**	1	3

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0	1	2	3	**4**
1	0	3	4	2
2	**3**	4	0	1
3	4	1	**2**	0
4	2	**0**	1	3

A transversal in a latin square of order q is a set of q entries, one selected from each row and each column such that no two entries contain the same symbol.

An n-ary quasigroup of order q is an n-ary operation $f : \Sigma_q^n \to \Sigma_q$ such that the equation $x_0 = f(x_1, \ldots, x_n)$ has a unique solution for any one variable if all the other n variables are specified arbitrarily.

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$$\begin{array}{cccc} 0&1&2&3\\ 1&0&3&2\\ 2&3&0&1\\ 3&2&1&0 \end{array} \times \begin{array}{cccc} 1&0&3&2\\ 0&1&2&3\\ 3&2&1&0\\ 2&3&0&1 \end{array} \times \begin{array}{cccc} 2&3&0&1\\ 3&2&1&0\\ 0&1&2&3\\ 1&0&3&2 \end{array} \times \begin{array}{cccc} 3&2&1&0\\ 2&3&0&1\\ 1&0&3&2\\ 0&1&2&3 \end{array}$$

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n-Ary quasigroups f and g are called *isotopic*, if there exists a collection of permutations $(\sigma_0, \sigma_1, \ldots, \sigma_n)$, $\sigma_i \in S_q$, such that

$$f(x_1, \ldots, x_n) = \sigma_0^{-1}\left(g(\sigma_1(x_1), \ldots, \sigma_n(x_n))\right).$$

A transversal in an n-ary quasigroup f of order q (or in an n-dimensional latin hypercube of order q) is a set of $(n+1)$-tuples

$$\left\{a^i = (a_0^i, a_1^i, \ldots, a_n^i)\right\}_{i=1}^{q}, \ a_k^i \in \Sigma_q$$

such that

$$a_0^i = f(a_1^i, \ldots, a_n^i), \text{ and } \rho(a^i, a^j) = n+1 \text{ for all } i \neq j,$$

where ρ is the Hamming distance.
Let $T(f)$ denote the number of transversals in a quasigroup f.

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$$\begin{matrix} \mathbf{0} & 1 & 2 & 3 \\ 1 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & \mathbf{0} \end{matrix} \times \begin{matrix} 1 & 0 & 3 & 2 \\ 0 & \mathbf{1} & 2 & 3 \\ 3 & \mathbf{2} & 1 & 0 \\ 2 & 3 & 0 & 1 \end{matrix} \times \begin{matrix} 2 & 3 & 0 & 1 \\ 3 & 2 & \mathbf{1} & 0 \\ 0 & 1 & \mathbf{2} & 3 \\ 1 & 0 & 3 & 2 \end{matrix} \times \begin{matrix} \mathbf{3} & 2 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & \mathbf{3} \end{matrix}$$

An n-ary quasigroup f of order q is called $\mathbb{Z}_q$-linear if it is isotopic to

$$g(x_1, \ldots, x_n) = x_1 + \ldots + x_n, \; x_i \in \mathbb{Z}_q.$$

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Statement

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Conjecture (Wanless, 2011)

Every latin hypercube of odd dimension or odd order has a transversal.

Transversals in completely reducible quasigroups

An n-ary quasigroup f of order q is a composition of an $(n-m+1)$-ary quasigroup h and an m-ary quasigroup g if there exists a permutation $\sigma \in S_n$ such that for all $x_1, \ldots, x_n \in \Sigma_q$

$$f(x_1, \ldots, x_n) = h(g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(n)}).$$

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An n-ary quasigroup f is called completely reducible if $n \leq 2$ or if for some $1 \leq m \leq n-1$ and for some permutation $\sigma \in S_n$ there exist a binary quasigroup h, a completely reducible m-ary quasigroup g, and a completely reducible $(n-m)$-ary quasigroup t such that

$$f(x_1, \ldots, x_n) = h(g(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), t(x_{\sigma(m+1)}, \ldots, x_{\sigma(n)})).$$

The quasigroup h in these decomposition is called external.

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Theorem 1

Let f be an n-ary completely reducible quasigroup of order q.

1. If n is odd then f has at least $(q \cdot q!)^{\frac{n-1}{2}}$ transversals.
2. If n is even and one of external quasigroups in a decomposition of f has a transversal, then f has at least $(q \cdot q!)^{\lfloor \frac{n-1}{2} \rfloor}$ transversals.

Let an n-ary quasigroup f of order q be a composition of an $(n-m+1)$-ary quasigroup h and an m-ary quasigroup g:

$$f(x_1, \ldots, x_n) = h(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_n).$$

Lemma 1

If the quasigroups g and h have $T(g)$ and $T(h)$ transversals respectively, then f has at least $T(g)T(h)$ transversals:

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$$f(x_1, \ldots, x_n) = h(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_n).$$

Lemma 2

Assume that for some $a \in \Sigma_q$

- the $(n-m)$-ary quasigroup h' defined by the equation $h(a, x_{m+1}, \ldots, x_n) = x_0$ has $T(h')$ transversals,
- the $(m-1)$-ary quasigroup g' defined by the equation $g(x_1, \ldots, x_m) = a$ has $T(g')$ transversals.

Then the quasigroup f has at least $q! \cdot T(h')T(g')$ transversals:

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Transversals in quasigroups of small order

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Statement

- If n is even then an n-ary quasigroup of order 2 has no transversals. If n is odd then an n-ary quasigroup of order 2 has 2^{n-1} transversals.
- An n-ary quasigroup of order 3 has $3^{n-2}(2^n - (-1)^n)$ transversals.

Let l be a function such that $l(0) = l(1) = 0,\ l(2) = l(3) = 1$.
An n-ary quasigroup f of order 4 is standardly semilinear if there exists a Boolean function λ such that $x_0 = f(x_1, \dots, x_n)$ iff

- $l(x_0) \oplus \dots \oplus l(x_n) = 0$;
- $x_0 \oplus \dots \oplus x_n \oplus \lambda(l(x_1), \dots, l(x_n)) = 0$.

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A quasigroup f is called semilinear if it is isotopic to some standardly semilinear quasigroup.

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Conjecture

If an n-ary quasigroup f of order 4 has no transversals then n is even and f is a $\mathbb{Z}_4$-linear quasigroup.

Thank you for your attention!