The Cayley Isomorphism Problem Exercises

1. Draw the Cayley graph $\mathrm{Cay}(\mathbb{Z}_9, \{1, 3, 6, 8\})$. Which group automorphisms of $\mathbb{Z}_9$ are contained in $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_9, \{1, 3, 6, 8\}))$?

2. Draw the Cayley graph $\mathrm{Cay}(Q_8, S)$. Here Q_8 is the **quaternion group** with presentation $\langle x, y : x^4 = y^4 = 1, x^2 = y^2, y^{-1}xy = x^{-1}\rangle$, and $S = \{x, y, x^{-1}, y^{-1}\}$.

3. Let Γ be a vertex-transitive digraph with Δ a subdigraph of Γ. Let $\mathcal{C}$ be the collection of all subdigraphs of Γ isomorphic to Δ. If $\mathcal{B} = \{V(C) : C \in \mathcal{C}\}$ is a partition of $V(\Gamma)$, then $\mathcal{B}$ is a complete block system of $\mathrm{Aut}(\Gamma)$.

4. Let $G \leq S_n$ be transitive with blocks B and B' such that $B \cup B'$ has order n. Show that $\{B, B'\}$ is a complete block system of G consisting of 2 blocks of size $n/2$.

5. Let $G \leq S_{np}$ have a complete block system $\mathcal{B}$ with blocks of prime size p where $p > n$. Show that $\mathcal{B}$ is a normal complete block system of G.

6. Show that a 2-transitive group is primitive.

7. Show that $\mathbb{Z}_4$ is a CI-group.

8. Are the digraphs $\mathrm{Cay}(\mathbb{Z}_{17}, \{1, 3, 5, 7, 11\})$ and $\mathrm{Cay}(\mathbb{Z}_{17}, \{3, 4, 10, 11, 12\})$ isomorphic?

9. Are the digraphs $\mathrm{Cay}(\mathbb{Z}_{13}, \{2, 5, 7, 9\})$ and $\mathrm{Cay}(\mathbb{Z}_{13}, \{1, 3, 5, 12\})$ isomorphic?

10. Show that for a transitive group $G \leq S_n$, its centralizer in S_n is semiregular. That is, the only permutation which fixes a point is the identity.

11. Prove

    Lemma 1 *Let* $\bar{A} \leq \mathrm{Aut}(G)$ *consist of all automorphisms of* G *that map* H *to* H. *Then* $N_{S_n}(G) = \bar{A} \cdot G$.