The Cayley Isomorphism Problem Second Exercises

1. Let Γ be a connected Cayley graph of a group G of order n and let p be the smallest prime divisor of n. Show that if Γ is regular of degree $d < p$, then Γ is a CI-graph of G.

2. Let Γ be a connected Cayley digraph of a nilpotent group G of order n, and let π be the set of primes dividing n. Show that if G is a Hall π-subgroup of $\mathrm{Aut}(\Gamma)$, then Γ is a CI-digraph of G.

3. Let $q < p$ be prime, and $\delta : \mathbb{Z}_{qp} \mapsto \mathbb{Z}_{qp}$ be given by $\delta(i, j) = (i, n_i j)$. Suppose that $\delta(\Gamma)$ is a CI-digraph of G, $G = \langle (\mathbb{Z}_{qp})_L, \delta^{-1}(\mathbb{Z}_{qp})_L \rangle$ has a complete block system $\mathcal{B}$ with blocks of size p, and that a Sylow p-subgroup of G has order p. Show that $\delta^{-1}(\mathbb{Z}_{qp})_L\delta = (\mathbb{Z}_{qp})_L$.

   The following problems are designed to investigate the Cayley isomorphisms problem for the nonabelian group G of order qp, where $q < p$ and $q|(p-1)$. Much is the same as in the case for $\mathbb{Z}_{qp}$, and we will assume:

   (a) Let $\alpha \in \mathbb{Z}_p^*$ of order q. We assume G permutes the set $\mathbb{Z}_q \times \mathbb{Z}_p$ and is generated by ρ and τ, where $\tau(i, j) = (i + 1, \alpha j)$ and $\rho(i, j) = (i, j + 1)$.

   (b) $\Gamma = \mathrm{Cay}(G, S)$, $H = \langle G_L, \delta^{-1}G_L\delta \rangle$ has a normal invariant partition with blocks of size p, a Sylow p-subgroup of $\mathrm{fix}_H(\mathcal{B})$ has order p, and $\delta(i, j) = (mi, n_i j + b_i)$.

   (c) Γ cannot be written as a nontrivial wreath product of a circulant of order q and a circulant of order p.

4. Show that if $\mathrm{Aut}(\Gamma)$ has a Sylow q-subgroup of order q, then Γ is a CI-digraph of G.

5. Show that $n_i = n_j$ for every $i, j \in \mathbb{Z}_q$.

6. Show that if $n_i = n_j = 1$, each $b_i = 0$, and $m \neq 1$, then the map $\bar{\alpha} : \mathbb{Z}_q \times \mathbb{Z}_p \mapsto \mathbb{Z}_q \times \mathbb{Z}_p$ be given by $\bar{\alpha}(i, j) = (i, \alpha j)$ is contained in $\mathrm{Aut}(\Gamma)$. Conclude that in this case Γ is a circulant digraph of order qp without relabeling!

7. Show that if $m = 1$ and $n_i = n_j = 1$ then $\delta \in \langle \rho \rangle \cdot \mathrm{Aut}(G)$.

8. Show that if Γ is not a CI-digraph of G, then Γ is also a circulant digraph of order qp.