The Cayley Isomorphism Problem Second Exercises

1. Let $\Gamma$ be a connected Cayley graph of a group $G$ of order $n$ and let $p$ be the smallest prime divisor of $n$. Show that if $\Gamma$ is regular of degree $d < p$, then $\Gamma$ is a CI-graph of $G$.

2. Let $\Gamma$ be a connected Cayley digraph of a nilpotent group $G$ of order $n$, and let $\pi$ be the set of primes dividing $n$. Show that if $G$ is a Hall $\pi$-subgroup of $\text{Aut}(\Gamma)$, then $\Gamma$ is a CI-digraph of $G$.

3. Let $q < p$ be prime, and $\delta : \mathbb{Z}_{qp} \rightarrow \mathbb{Z}_{qp}$ be given by $\delta(i, j) = (i, n_i j)$. Suppose that $\delta(\Gamma)$ is a CI-digraph of $G$, $G = \langle (\mathbb{Z}_{qp})_L, \delta^{-1}(\mathbb{Z}_{qp})_L \rangle$ has a complete block system $\mathcal{B}$ with blocks of size $p$, and that a Sylow $p$-subgroup of $G$ has order $p$. Show that $\delta^{-1}(\mathbb{Z}_{qp})_L \delta = (\mathbb{Z}_{qp})_L$.

The following problems are designed to investigate the Cayley isomorphisms problem for the nonabelian group $G$ of order $qp$, where $q < p$ and $q | (p - 1)$. Much is the same as in the case for $\mathbb{Z}_{qp}$, and we will assume:

(a) Let $\alpha \in \mathbb{Z}_p^*$ of order $q$. We assume $G$ permutes the set $\mathbb{Z}_q \times \mathbb{Z}_p$ and is generated by $\rho$ and $\tau$, where $\tau(i, j) = (i + 1, \alpha j)$ and $\rho(i, j) = (i, j + 1)$.

(b) $\Gamma = \text{Cay}(G, S)$, $H = \langle G_L, \delta^{-1}G_L \delta \rangle$ has a normal invariant partition with blocks of size $p$, a Sylow $p$-subgroup of $\text{fix}_H(\mathcal{B})$ has order $p$, and $\delta(i, j) = (m_i, n_i j + b_i)$.

(c) $\Gamma$ cannot be written as a nontrivial wreath product of a circulant of order $q$ and a circulant of order $p$.

4. Show that if $\text{Aut}(\Gamma)$ has a Sylow $q$-subgroup of order $q$, then $\Gamma$ is a CI-digraph of $G$.

5. Show that $n_i = n_j$ for every $i, j \in \mathbb{Z}_q$.

6. Show that if $n_i = n_j = 1$, each $b_i = 0$, and $m \neq 1$, then the map $\bar{\alpha} : \mathbb{Z}_q \times \mathbb{Z}_p \mapsto \mathbb{Z}_q \times \mathbb{Z}_p$ be given by $\bar{\alpha}(i, j) = (i, \alpha j)$ is contained in $\text{Aut}(\Gamma)$. Conclude that in this case $\Gamma$ is a circulant digraph of order $qp$ without relabeling!

7. Show that if $m = 1$ and $n_i = n_j = 1$ then $\delta \in \langle \rho \rangle \cdot \text{Aut}(G)$.

8. Show that if $\Gamma$ is not a CI-digraph of $G$, then $\Gamma$ is also a circulant digraph of order $qp$. 