Minimal supports of eigenfunctions of Hamming graphs

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Many combinatorial configurations (for example, perfect codes, latin squares and hypercubes, combinatorial designs and their q-ary generalizations — subspace designs) can be defined as an eigenfunction on a graph with some discrete restrictions. The study of these configurations often leads to the question about the minimum possible difference between two configurations from the same class (it is often related with bounds of the number of different configurations; for example, see [1–5]). Since the symmetric difference of these two configurations is also an eigenfunction, this question is directly related to the minimum cardinality of the support (the set of nonzero) of an eigenfunction with given eigenvalue. This paper is devoted to the problem of finding the minimum cardinality of the support of eigenfunctions in the Hamming graphs $H(n, q)$. Currently, this problem is solved only for $q = 2$ (see [4]). In [6] Vorob'ev and Krotov proved the lower bound on the cardinality of the support of an eigenfunction of the Hamming graph. In this paper we find the minimum cardinality of the support of eigenfunctions in the Hamming graphs with eigenvalue $n(q-1)-q$ and describe the set of functions with the minimum cardinality of the support.

It is well-known that the set of eigenvalues of the adjacency matrix of $H(n, q)$ is $\{\lambda_m = n(q-1) - qm \mid m = 0, 1, \ldots, n\}$. The support of f is denoted by $S(f)$. The set of vertices $x = (x_1, x_2, \ldots, x_n)$ of the graph $H(n, q)$ such that $x_i = k$ is denoted by $T_k(i, n)$. We prove the following theorem:

Theorem. *Let $f : H(n, q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to λ_1, $f \not\equiv 0$ and $q > 2$. Then $|S(f)| \geq 2(q-1)q^{n-2}$. Moreover, if $|S(f)| = 2(q-1)q^{n-2}$, then*

$$f(x) = \begin{cases} c, & \text{for } x \in T_k(i, n) \setminus T_m(j, n); \\ -c, & \text{for } x \in T_m(j, n) \setminus T_k(i, n); \\ 0, & \text{otherwise;} \end{cases}$$

where $c \neq 0$ is a constant, i, j, k, m are some numbers and $i \neq j$.

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