Majorana representations of triangle-point groups

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- If $t, s \in 2A$ then ts is of order at most 6 and belongs to one of nine conjugacy classes:

$$1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A.$$

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- The 2A-axes generate the Griess algebra i.e. $V_{\mathbb{M}} = \langle\langle \psi(t) : t \in 2A \rangle\rangle$
- If $t, s \in 2A$ then the algebra $\langle\langle \psi(t), \psi(s) \rangle\rangle$ is called a dihedral subalgebra of $V_{\mathbb{M}}$ and has one of nine isomorphism types, depending on the conjugacy class of ts.

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The algebra V also contains the axis $\psi(ts)$. In fact, it is of dimension 3:

$$V = \langle \psi(t), \psi(s), \psi(ts) \rangle_{\mathbb{R}}.$$

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- The central object in his proof is the Moonshine module, denoted $V^{\#}$. It belongs to a class of graded algebras know as vertex operator algebras, or VOA's
- In particular, we have $\mathrm{Aut}(V^{\#}) = \mathbb{M}$

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- If we take a vertex operator algebra

$$V = \bigoplus_{n=0}^{\infty} V_n$$

such that $V_0 = \mathbb{R}\mathbf{1}$ and $V_1 = 0$ then V_2 is a real, commutative, non-associative algebra called a generalised Griess algebra

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- If $V = V^{\#}$, then $V_2 \cong V_{\mathbb{M}}$, the τ_a are the 2A involutions and the $\frac{1}{2}a$ are the 2A axes.

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Theorem (S. Sakuma, 2007)

Any subalgebra of a generalised Griess algebra generated by two Ising vectors is isomorphic to a dihedral subalgebra of the Griess algebra.

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- The Griess algebra $V_{\mathbb{M}}$ is an example of a Majorana algebra, with the 2A involutions and 2A axes corresponding to Majorana involutions and Majorana axes respectively
- Almost all known Majorana algebras occur as subalgebras of $V_{\mathbb{M}}$.

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  - V is a Majorana algebra
  - φ is a homomorphism $G \to GL(V)$
  - $\psi : T \to V \backslash \{0\}$ is an injective mapping such that $\psi(t^g) = \psi(t)^{\varphi(g)}$ and such that $\psi(t)$ is a Majorana axis for all $t \in T$.

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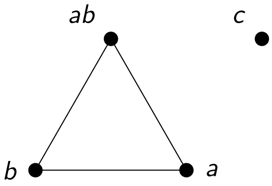

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Theorem (S. Decelle, 2012)

If G is a triangle-point group then it must occur as the quotient of one of eleven groups, all of which are finite.

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Theorem (W. 2016)

There are at most 7 pairwise non-isomorphic triangle-point groups which admit a Majorana representation (G, T, V, ϕ, ψ) such that

$$a^G \cup b^G \cup c^G \cup (ab)^G \subseteq T.$$

but which do not occur as a triangle-point configurations in the Monster graph.