On Eulerian Identities in $UT_n(\mathbb{Q})$

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One of the known methods of building identities in matrix rings deals with eulerian multigraphs. A multigraph is a graph which is permitted to have multiple oriented edges with same end nodes, and loops - edges with same start and end node. Suppose, $G$ is an eulerian graph with $n$ edges and $\Pi(G)$ is the set of all its eulerian paths. Every such path $\pi$ we consider to be a permutation of set $\{1, 2, \ldots, n\}$, by $\text{sgn}(\pi)$ we denote the sign of this transposition. An eulerian polynomial of the graph $G$ is the polynomial:

$$P_G = \sum_{\pi \in \Pi(G)} \text{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(n)}.$$  

The examples of eulerian polynomials are well known standard polynomials

$$S_n = \sum_{\pi \in S_n} \text{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(n)},$$

which are built from graphs with one vertex and $n$ loops. The identity $f = 0$ is called eulerian, if $f$ is an eulerian polynomial of some graph.

The identity basis of $2 \times 2$ matrices ring from eulerian polynomials is already found by M. Domokos in [1]. The following result deals with ring of upper triangular matrices $UT_n(\mathbb{Q})$, the identity basis of such ring from non-eulerian polynomials is also found by Yu. N. Maltsev in [2].

**Theorem.** In the class of rings with 1, the eulerian polynomial

$$P_{UT_n}(x_1, \ldots, x_n, y_1, \ldots, y_n, e_1, \ldots, e_{n-1}) = [x_1, y_1][x_2, y_2]\cdots e_{n-1}[x_n, y_n]$$

forms basis of the ring $UT_n(\mathbb{Q})$. The polynomial $P_{UT_n}$ is built from graph $G_{UT_n}$ displayed on fig.1.

![Graph G_{UT_n}](image)

**Figure 1:** $G_{UT_n}$

**References**
