On the spectra of automorphic extensions of finite simple exceptional groups of Lie type

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$\omega(G)$ — the set of orders of the elements of G, or its spectrum

Groups are isospectral if their spectra coincide.

$h(G)$ — the number of pairwise non-isomorphic groups isospectral to G.

G is recognizable by its spectrum if $h(G) = 1$, i.e. for any group H

$$\omega(H) = \omega(G) \Rightarrow G \simeq H$$

Recognition by spectrum problem is solved for a group G if we know $h(G)$ (and if $h(G)$ is finite then the groups isospectral to G are determined).

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Main result

Recognition problem is solved for all simple exceptional groups of Lie type.

- 1992–1999, Brandl, Shi, Deng:
  $^2B_2(q)$, $^2G_2(q)$, $^2F_4(q)$ — recognizable
- 2002, Vasil'ev: $G_2(3^m)$ — recognizable
- 2005, Vasil'ev, Mazurov, Shi, ... : $F_4(2^m)$ — recognizable
- 2010, Kondrat'ev: $E_8(q)$ — recognizable
- 2013, Vasil'ev, Staroletov: $G_2(q)$ — recognizable

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Problem (16.24 Kourovka Notebook)

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- 2013, Mazurov: $h({}^3D_4(2)) = \infty$

Remaining groups: ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$

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Theorem A. Let S be a finite simple exceptional group of Lie type and $S \neq {}^3D_4(2)$. Then a finite group isospectral to S is isomorphic to a group G, such that $S \leq G \leq \operatorname{Aut} S$. In particular, $h(S)$ is finite.

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- 2005, Alekseeva, Kondrat'ev: ${}^3D_4(q)$, $F_4(q)$ — quasirecognizable
- 2007, Kondrat'ev: $E_6(q)$, ${}^2E_6(q)$ — quasirecognizable
- 2014, Vasil'ev, Staroletov: $E_7(q)$ — quasirecognizable
- 2015, Grechkoseeva: S is recognizable among covers

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- 2015, Grechkoseeva, Zvezdina: $^3D_4(q)$, $F_4(q)$
- 2016, Zvezdina: $E_6(q)$, $^2E_6(q)$, $E_7(q)$

New results

Notation: $E_6^+(q) = E_6(q)$ and $E_6^-(q) = {}^2E_6(q)$ are denoted by $E_6^\varepsilon(q)$, $\varepsilon \in \{+, -\}$.

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Theorem 1. Let $S = E_6^\varepsilon(q)$, where q is a power of a prime p, and $S < G \leq \operatorname{Aut} S$. Then $\omega(G) = \omega(S)$ if and only if G is an extension of S by a field automorphism, G/S is a 3-group, 3 divides $q - \varepsilon 1$, and $p \notin \{2, 11\}$.

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Example. If $S = E_6(5^6)$, $S < G \leq \operatorname{Aut} S$ and $\omega(G) = \omega(S)$, then $G \simeq S \rtimes \langle \varphi \rangle$, where φ is a field automorphism of S of order 3. In particular, $h(S) = 2$.

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Theorem 2. Let $S = E_7(q)$, where q is a power of a prime p, and $S < G \leq \operatorname{Aut} S$. Then $\omega(G) = \omega(S)$ if and only if G is an extension of S by a field automorphism, G/S is a 2-group, and $p \notin \{2, 13, 17\}$.

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Theorem B. Let S be a simple exceptional group of Lie type ${}^dX_n(q)$, where $q = p^m$, p is a prime. Then $h(S)$ is as indicated in Table 1. If $1 < h(S) < \infty$, then a finite group is isospectral to S if and only if it is isomorphic to a group G such that $S \leq G \leq S \rtimes \langle \varphi \rangle$, where φ is a field automorphism of a group S of the order given in Table 1.

$(m)_r$ is the largest power of a prime r dividing an integer m.

Table 1

S	Conditions	$\lvert\varphi\rvert$	$h(S)$
${}^2B_2(q)$		$-$	1
${}^2G_2(q)$		$-$	1
${}^2F_4(q)$		$-$	1
$G_2(q)$		$-$	1
$E_8(q)$		$-$	1
${}^3D_4(q)$	$p \notin \{2,3,7,11\}$, $(m)_2 = 2^s > 2$	2^s	$s+1$
	($p \in \{2,3,7,11\}$ or m is odd) and $q \neq 2$	$-$	1
	$q = 2$	$-$	∞
$F_4(q)$	$p \notin \{2,3,7,11\}$, $(m)_2 = 2^s > 2$	2^s	$s+1$
	otherwise	$-$	1
$E_6^{\varepsilon}(q)$	$p \notin \{2,11\}$, $3 \mid q-\varepsilon 1$, $(m)_3 = 3^s > 3$	3^s	$s+1$
	otherwise	$-$	1
$E_7(q)$	$p \notin \{2,13,17\}$, $(m)_2 = 2^s > 2$	2^s	$s+1$
	otherwise	$-$	1