Intersection of conjugate solvable subgroups in classical groups of Lie type

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Definitions

Assume a finite group G acts transitively on a set Ω, (i.e. $\forall x, y \in \Omega$ exist $g \in G : (x)g = y$) and K is the kernel of this action (i.e. $K = \{g \in G : \forall x \in \Omega, (x)g = x\}$) . G_x is the stabilizer of a point x,
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Assume that for $x_1, \dots, x_m \in \Omega$, inequality $G_{x_1} \cap \dots \cap G_{x_m} = K$ holds. Then the action of $g \in G$ on Ω is completely determined by the action on $x_1, \dots, x_m$, specifically, if g stabilazes $x_1, \dots, x_m$, then $g \in K$.

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Assume that for $x_1, \dots, x_m \in \Omega$, inequality $G_{x_1} \cap \dots \cap G_{x_m} = K$ holds. Then the action of $g \in G$ on Ω is completely determined by the action on $x_1, \dots, x_m$, specifically, if g stabilazes $x_1, \dots, x_m$, then $g \in K$.
If $\{x_1, \dots, x_m\}$ has the minimal cardinality then it is called a base of G and is denoted by $Base(G) = \{x_1, \dots, x_m\}$. The size of a base of G is denoted by $b(G)$.

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Then $v_1, \ldots, v_n$ is a base in our definition if and only if it is a basis of vector space Ω.
Thus in the case of vector spaces we can reduce the study of group G to the study of $n \times n$ matrix by using a base. This leads to the possibility of algorithmic solution to many complicated problems.

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Since for $x, y \in \Omega$, and $g \in G$ with $(x)g = y$, we have $g^{-1}G_x g = G_y$, the stabilizers of points lying in the same orbit are conjugate. Thus the action does not depend on the choice of point stabilizer.
So in case of transitive action the base size for the group G and point stabilizer $G_x =: H$ is equal to the minimal number k of elements $g_1, \ldots, g_k \in G$ such that

$$H^{g_1} \cap \cdots \cap H^{g_k} = H_G,$$

(Here $H_G = \cap_{g \in G} H^g$ is the core of subgroup H.)

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If G acts faithfully and transitively on Ω then, as easy to see, the minimal number k such that the set Ω^k contains a G-regular point is the base size of G.
For a positive integer m the number of G-regular orbits on Ω^m is denoted by $Reg(G, m)$ (this number equals 0 if $m < b(G)$).

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In this case, we denote $b(G/H_G)$ and $Reg(G/H_G, m)$ by $b_H(G)$ and $Reg_H(G, m)$ respectively.
Thus $b_H(G)$ is the minimal number k such that there exist elements $x_1, \ldots, x_k \in G$ with $H^{x_1} \cap \ldots \cap H^{x_k} = H_G$.

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“Kourovka notebook”, 17.41(b)

Let H be a solvable subgroup of a finite group G and G does not contain nontrivial normal solvable subgroups. Are there always exist five subgroups conjugate with H such that their intersection is trivial, i.e. do there exist $g_1, g_2, g_3.g_4 \in G$ such that the identity

$$H \cap H^{g_1} \cap H^{g_2} \cap H^{g_3} \cap H^{g_4} = 1$$

holds?

Previously known results

The problem is reduced to the case when G is almost simple in

[1] *E. P. Vdovin*, On the base size of a transitive group with solvable point stabilizer, Journal of Algebra and Application, v. 11 (2012), N 1, 1250015 (14 pages).

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and G_0 is nonabelian simple group.
Specifically, it is proved that if for each almost simple group G and solvable subgroup H of G condition $Reg_H(G,5) \geqslant 5$ holds then for each finite nonsolvable group G and solvable subgroup H of G condition $Reg_H(G,5) \geqslant 5$ holds.

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A.B., 2015

Let H be a solvable subgroup of an almost simple group G whose socle is isomorphic to A_n, $n \geqslant 5$. Then $Reg_H(G,5) \geqslant 5$. In particular $b_H(G) \leqslant 5$.

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Tim Burness, 2007

Let G be a finite almost simple classical group in a faithful primitive non-standard action. Then either $b(G) \leqslant 4$, or $G = U_6(2) \cdot 2$, $H = U_4(3) \cdot 2^2$ and $b(G) = 5$.

The main result

Let p be a prime number and $q = p^t$. A cyclic irreducible subgroup $Sin_n(q)$ of $GL_n(q)$ of order $q^n - 1$ is called a *Singer cycle*. If H is a cycle subgroup of $GU_n(q)$ and $|H| = q^n - (-1)^n$ we also call it a Singer cycle and denote by $Sin_n(q)$.

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By φ_n we denote an automorphism of $Sin_n(q)$ such that $\varphi_n : g \mapsto g^q$ if $G = GL_n(q)$ and $\varphi_i : g \mapsto g^{q^2}$ if $G = GU_n(q)$.

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Theorem 1

Let G be isomorphic to $GL_n(q)$ or $GU_n(q)$ and H be a subgroup of G such that H is block diagonal with blocks isomorphic to $Sin_{n_i}(q) \rtimes \langle \varphi_{n_i} \rangle; i = 1, \ldots, k; \sum_{i=1}^{k} n_i = n$. Then $b_H(G) \leqslant 4$.

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Theorem 2

Let $G = GL_n(q) \rtimes \langle \tau \rangle$ where $q = 2$ or $q = 3$, n is even, τ is an automorphism which acts by $\tau : A \mapsto (A^{-1})^T$ for $A \in GL_n(q)$. Let H be the normalizer in G of subgroup $P \leqslant GL_n(q)$ where P is the stabilizer of the chain of subspaces:

$$\langle v_n, v_{n-1} \rangle < \langle v_n, v_{n-1}, v_{n-2}, v_{n-3} \rangle < \ldots < \langle v_n, v_{n-1}, \ldots, v_2, v_1 \rangle.$$

Then $Reg_H(G, 5) \geqslant 5$.

THANK YOU FOR ATTENTION!