HAMILTON CYCLES IN MAPS

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joint work with
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Motivation: Graph symmetry and hamiltonicity

General problem.
Which combinatorial properties of graphs are implied by graph symmetry?

Question (Lovász, 1969)

Does every connected vertex-transitive graph have a Hamilton path, i. e., a simple path going through all vertices?

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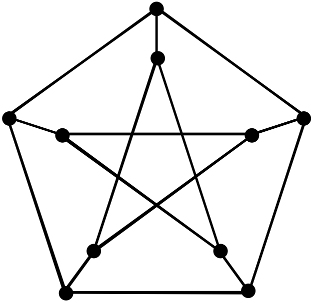
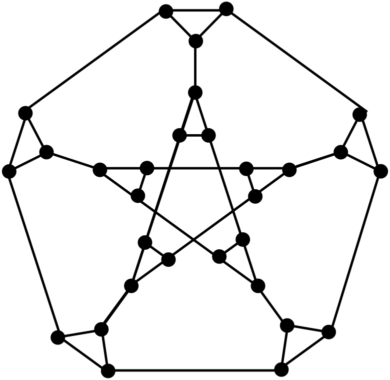
Only five connected vertex-transitive graphs with no Hamilton cycle are known (but they do have a Hamilton path), namely, K_2, Petersen graph, Coxeter graph, the truncations of the Petersen and Coxeter graphs

Non-hamiltonian cubic vertex-transitive graphs





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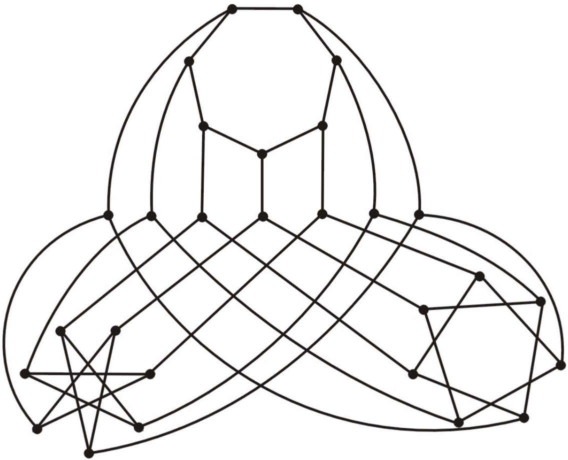

Hamilton cycles in Cayley graphs

All known non-hamiltonian vertex-transitive graphs are cubic, and none of them is a Cayley graph. There are different opinions about the hamiltonicity of vertex-transitive graphs!

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Counter-conjecture (Babai, 1995)

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Conjecture (Folklore)

Every Cayley graph (of order ≥ 3) has a Hamilton cycle.

Hamilton cycles in Cayley graphs: current status

Hamilton cycles or paths exist, if strong restrictions on the structure of the base groups are introduced, for instance: in the following types of vertex-transitive graphs:

- Graphs of specific order:
  p, $2p$, $3p$, $4p$, $5p$, $6p$, $2p^2$, ... (where p a prime)
- Cayley graphs of p-groups
- Cayley graphs of groups with a cyclic commutator subgroup of order p^k.

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Until 2009, very little was known about Hamilton cycles or paths in cubic vertex-transitive graphs including Cayley cubic graphs.

Hamiltonicity of cubic graphs in general?

- To decide whether a cubic graph is hamiltonian is an NP-complete problem even if we are restricted to planar cubic graphs,
- Hamiltonicity implies 3-edge colourability, there infinitely many non-3-edge-colourable cubic graphs, called snarks. These are by the four colour theorem non-planar.
- All known cubic non-hamiltonian graphs have cyclic-connectivity at most 7. Only one cyclically 7-connected example is known.
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- Thomassen's conjecture: Cubic graphs with sufficiently large cyclic connectivity (≥ 8) are hamiltonian!
- Known technique: lifting Hamilton cycle along a graph covering.
- In what follows we analyse a new technique to prove hamiltonicity introduced in 2009 by Glover and Marušič.

Cubic Cayley graphs

There are of two types: if $X = Cay(G; S)$, then either

1. $S = \{x, y, z\}$, where $x^2 = y^2 = z^2 = 1$ are involutions, or
2. $S = \{x, y\}$, where $x^2 = 1$ and $|x| > 2$.

Theorem by Glover & Marušič

Let $H = \langle r, l \rangle$ be a $(2, 3, s)$-presented finite group; i.e., $r^s = l^2 = (rl)^3 = 1$.

Then H is a finite quotient of the modular group $PSL(2, \mathbb{Z})$.

Theorem (Glover & Marušič, 2009)

Let $K = \mathrm{Cay}(H; r, r^{-1}, l)$ be a cubic Cayley graph, where $H = \langle r, l \mid r^s = l^2 = (rl)^3 = 1, \ldots \rangle$ is a finite quotient of the modular group $PSL(2, \mathbb{Z})$. Then K has a Hamilton path. Moreover,

- *if $|H| \equiv 2 \pmod 4$, then K has a Hamilton cycle*
- *if $|H| \equiv 0 \pmod 4$, then K has a cycle through all but two adjacent vertices.*

Sample of new results: Cayley graphs

Theorem (I.)

Let $G = \langle x, y | y^2 = (xy)^3 = 1, \dots \rangle$. *Then* $Cay(G; x, x^{-1}, y)$ *admits a bounding H.C. with respect to standard embedding if and only if* $|G| \equiv 2$ *mod* 4, *or* $|x| \equiv 0, 1, 3$ *mod* 4.

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Theorem (II.)

Let $G = \langle x, y, z | x^2 = y^2 = z^2 = (xy)^3 = (yz)^3 = 1, \dots \rangle$*. Then* $Cay(G; x, y, z)$ *admits a contractible H.C. with respect to standard embedding if* $|G| \equiv 2$ *mod* 4*, and a contractible cycle missing two adjacent vertices otherwise.*

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$|G| = 6(b^2 + bc + c^2)$ thus $|G| \equiv 0 \mod 4$ iff both b and c are even,
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Altshuler: All Coxeter maps are hamiltonian, thus in case b and c is even, all H.C. must be unbounding!

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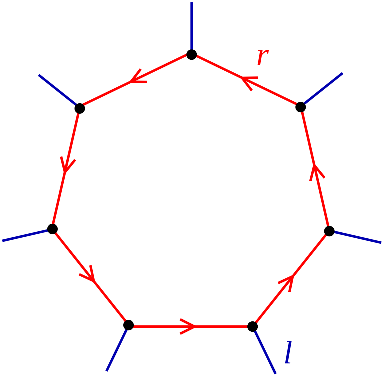
Theorem (C)

(Dipendu Maity) Triangulations with constant degree of vertices are hamiltonian.

Method: Standard embedding of a cubic Cayley graph



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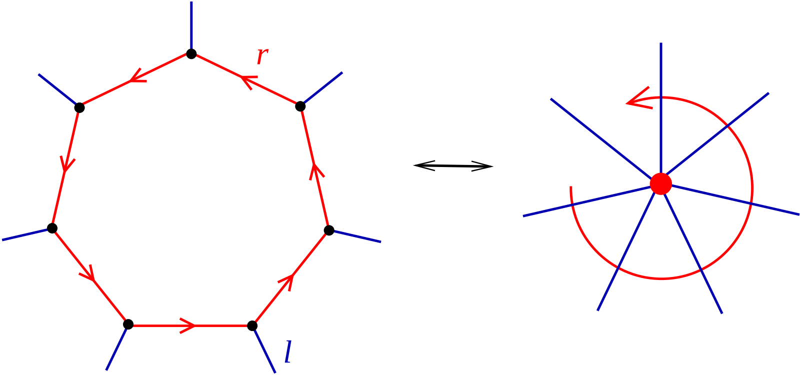

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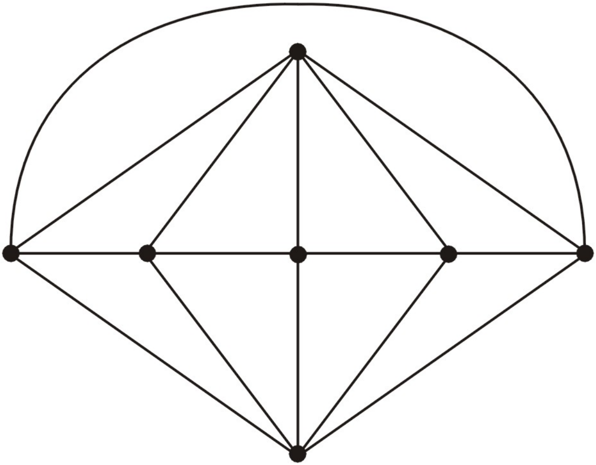
3 Boundary $\partial(\cup_{F \in A} \bar{F})$ of the topological closure of the union of faces in A forms a bounding hamilton cycle.

Example: Construction of a Hamilton cycle in $t(\mathcal{T})$



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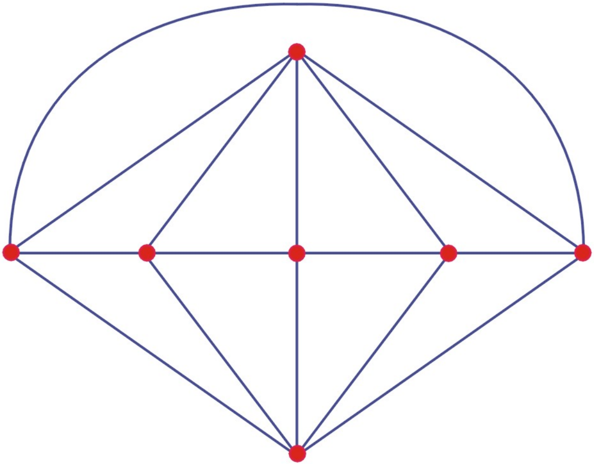

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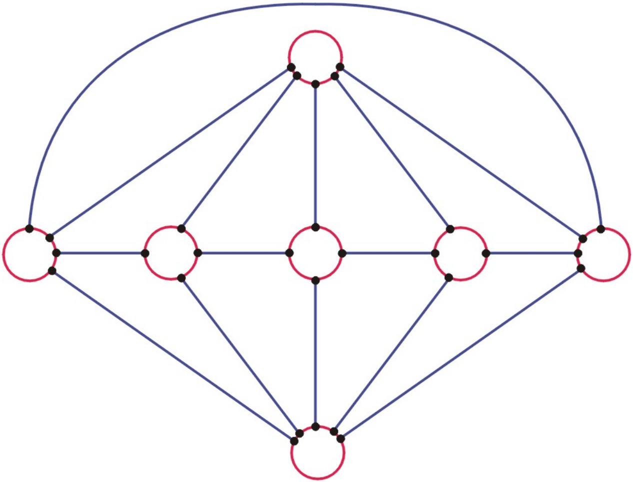

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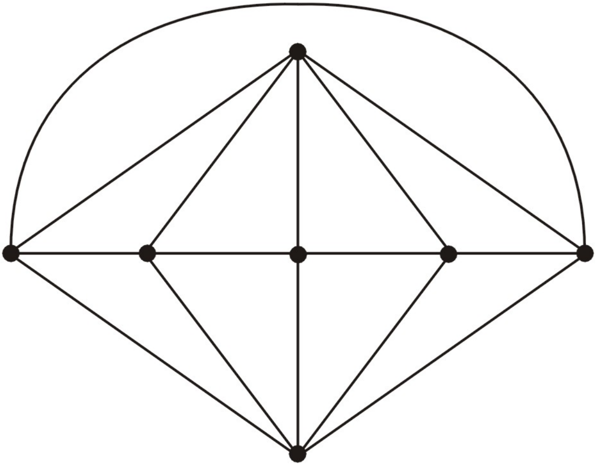

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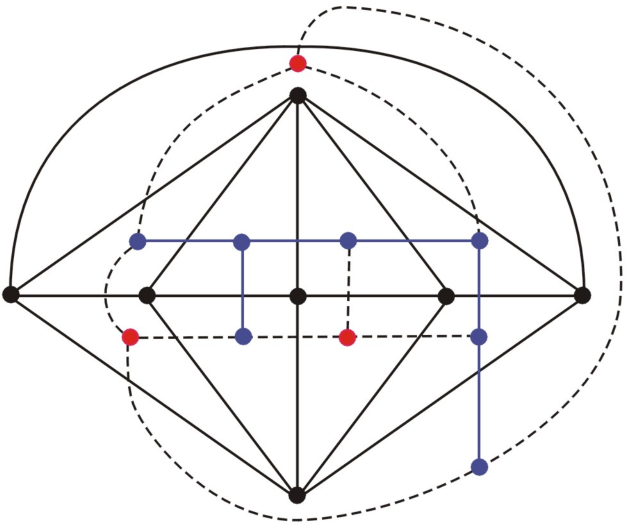

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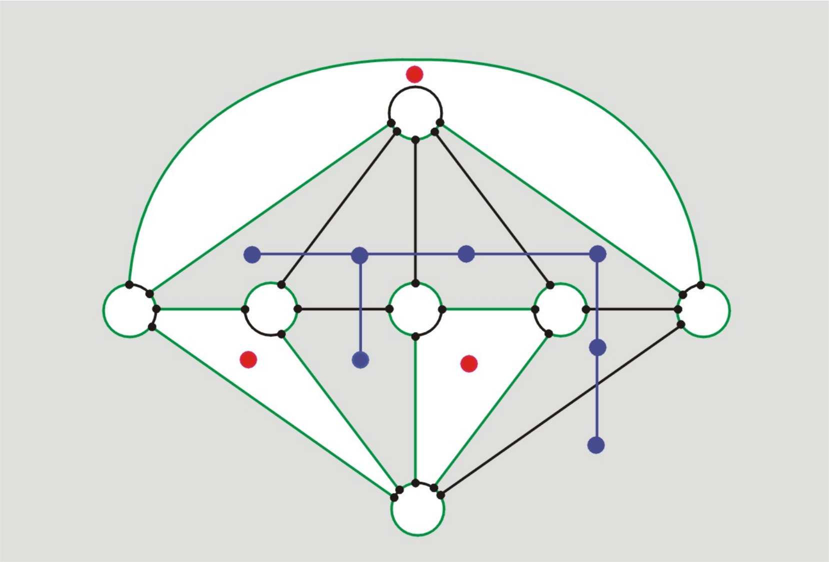
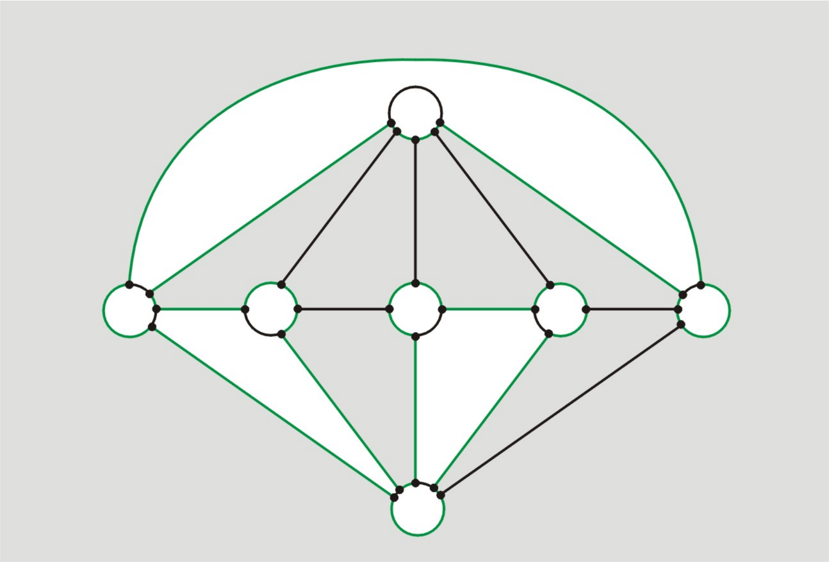
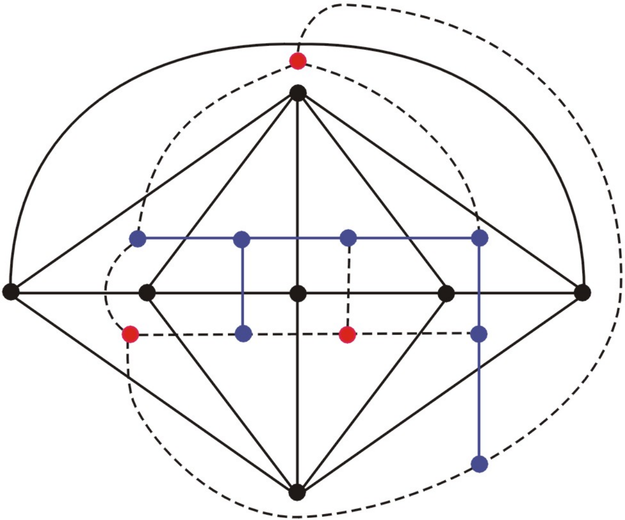


Example: The required Hamilton cycle



When does such a structure exist?



Two important concepts

1−sided subgraph in the dual of a map, an induced subgraph embedded with a connected complement, generalisation of a plane tree from the spherical case,

weak 2-face colouring = colouring of faces of a (cubic) map by two colours such every vertex is incident to faces of two colour,

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This introduces the separation property!

H.C. in embedded graphs: investigate the dual!

The following theorem generalises an observation from the spherical case: A sperical map admits H.C. iff the dual admits a decomposition into two induced trees.

Theorem

The following statements are equivalent for a polytopal map $\mathcal{M}$ on a closed surface S.

(i) *$\mathcal{M}$ has a bounding Hamilton cycle.*

(ii) *The vertex set of $\mathcal{M}^*$ admits a partition into two subsets which induce one-sided subgraphs H and K such that $\beta(H) + \beta(K) = \epsilon(S)$.*

(iii) *$\mathcal{M}$ has a weak 2-face-colouring ϕ such that the vertices of $\mathcal{M}^*$ that receive colour 1 from ϕ induce a one-sided subgraph of $\mathcal{M}^*$.*

Furthermore, if (iii) is fulfilled, then the set of faces of colour 1 is bounded by a Hamilton cycle.

Weak 2-face colouring - tool to find co-hamiltonian decomposition

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(iii) *If the set of 1-vertices of $\mathcal{M}^*$ has a partition $\{A, J\}$ such that A induces a forest with two components and J is independent, then $\mathcal{M}$ has a 2-factor with two components.*

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In cases (ii) *and* (iii)*, the underlying graph of $\mathcal{M}$ has a Hamilton path.*

Truncated maps = maps with a particular 2-face colouring

Truncation: Let M be a map on a closed surface, orientable, or not. By a truncation $t(M)$ we mean the unique cubic map which arises from M by expanding every vertex v of M to a cycle of length $deg(v)$.

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Observation: A cubic map N is a truncation of a map M if and only if we can colour faces of N by two coloures such that one-coloured faces give rise to a facial 2-factor, a 2-factor where each component bounds a face.

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Standard embedding: If $X = Cay(G; x, x^{-1}, y)$ is a Cayley cubic graph we form the standard embedding $X \hookrightarrow S$ by gluing a face to the 2-factor coloured by x, and by gluing a face to each cycle coloured alternatingly by x and y. The standard embedding is the truncation of a regular map on an orientable surface! We have a particular weak 2-face-colouring

Vertex-bipartitions of the dual of a triangulation

Theorem (Payan & Sakarovitch, 1975)

Let G be a cyclically 4-edge-connected cubic graph with n vertices. Then the following hold:

(i) *If $n \equiv 2 \pmod 4$, then $V(G)$ has a partition $\{A, J\}$ where A induces a tree and J is independent.*

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Case (i) implies a Hamilton cycle in $\mathrm{Cay}(H; r, r^{-1}, l)$

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Show idea of the proof!

Corollaries (samples)

Theorem (K., N. & S., 2013+)

Let $\mathcal{T}$ be a triangulation of a closed surface by f triangles which is either edge-transitive or has no separating cycle of length ≤ 3. Then $t(\mathcal{T})$ has a Hamilton path. Moreover,

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Method of proof: Kempe switches on weak 2-face colourings!

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$\partial(T^*)$ is hamiltonian cycle of the halved fullerine!

Few words about the result by Dipendu Maity

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Archdeacon, Hardsfield, Little, J.Comb.Theory B 1996: There exists infinitely many non-hamiltonian triangulations of arbitrary large connectivity and arbitrary large planar width!

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6. Insert a new vertex inside every face and join it to all old vertices on the boundary! The triangulation T cannot be hamiltonian, since the new vertices form an independent set of size $> |V(T)|/2$.

Open problems

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2. In view of the Thomassen conjecture, try other families of cubic Cayley graph satisfying a short non-trivial relation.
3. Confirm the result on hamiltonicity of q-valent triangulations.

Thank you!