The four color theorem and Thompson's *F*

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Thompson's F

$\boldsymbol{Def}(\boldsymbol{Thompson's\ F})$

Condition Q

- φ: $[0,1] \rightarrow [0,1]$ is piecewise linear homeomorphism
- φ is differentiable except at finitely $\frac{b}{2^a}$ form numbers $(a, b \, \epsilon \mathbb{Z})$
- on differentiable interval of φ, the derivatives are powers of 2

$F \coloneqq \{\varphi \mid \varphi$ meets condition $Q\}$ is a group by composition of maps.

$$F \cong \langle A, B \mid [AB^{-1}, A^{-1}BA], [AB^{-1}, A^{-2}BA^2] \rangle$$

with $[x, y] = xyx^{-1}y^{-1}$

Cannon, J.W., Floyd, W.J., Parry, W.R.: Introductory notes on Richard Thompson's groups. Enseign. Math. (2) 42(3–4), 215–256 (1996)

Four color theorem

Four color theorem

Every planar graph has a face 4-coloring.

$\Leftrightarrow$

Let F be Thompson's F. $\forall f \in F, f$ is colorable.

By Bowlin and Brin, 2013

Binary trees

***Def* (Binary tree)**
$\{0,1\}^* := \{\textit{finite words in the alphabets } 0 \textit{ and } 1\} \cup \{\emptyset\}$

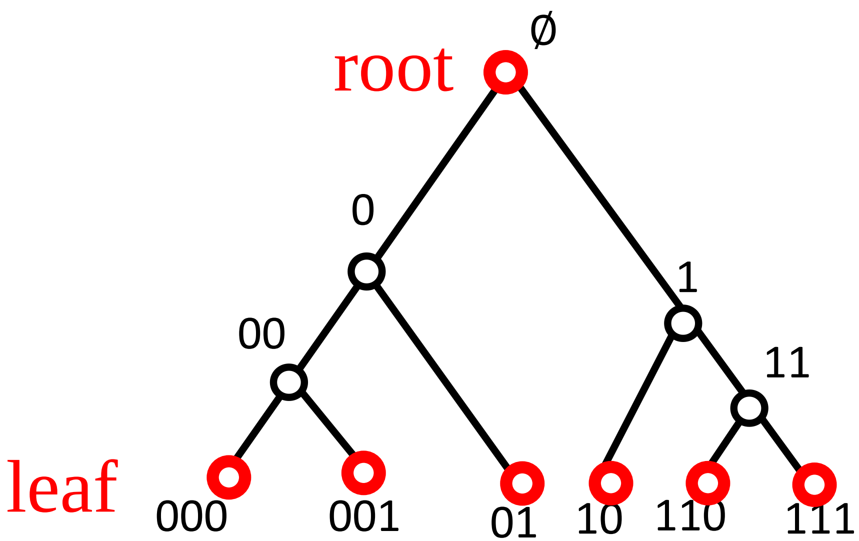
If a finite set G satisfies these conditions as follows

1. $G \subset \{0,1\}^*, \emptyset \in G,$

2. $\forall w \in G, (w0 \in G \land w1 \in G) \lor (w0 \notin G \land w1 \notin G),$

3. $w0 \in G \lor w1 \in G \Rightarrow w \in G,$

then we say that G is a **binary tree**.



Ex: $G = \{\emptyset, 0, 1, 00, 01, 000, 001, 10, 11, 110, 111\}$

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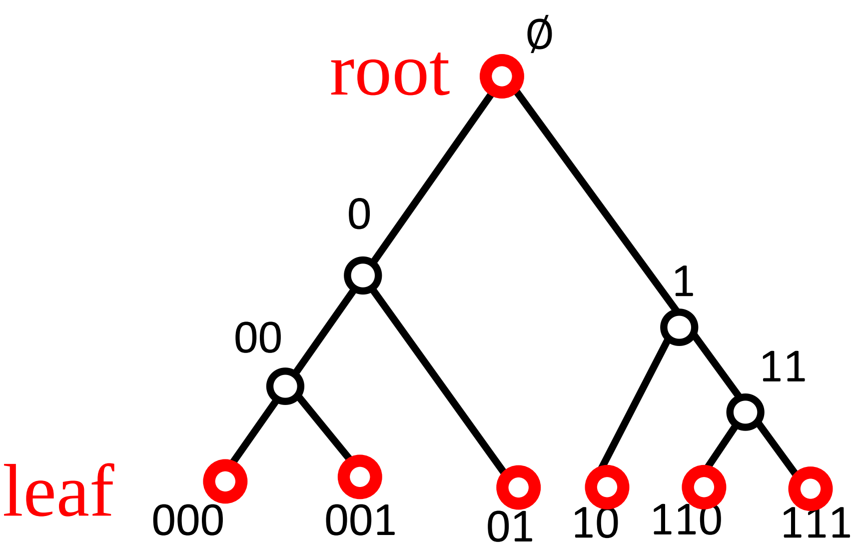
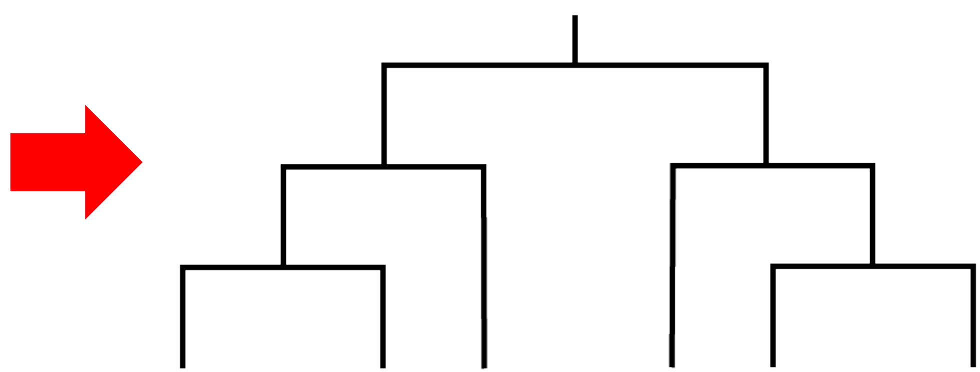
We should regard binary trees as knockout tournaments.



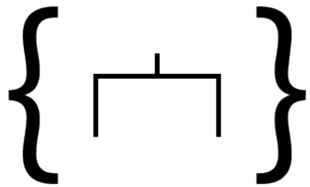


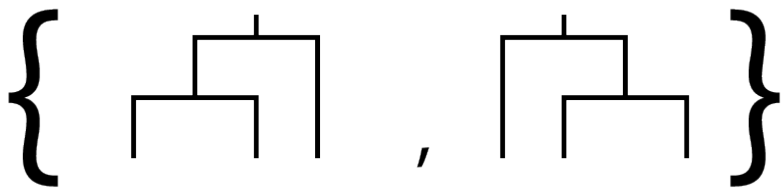
Ex: $G = \{\emptyset, 0, 1, 00, 01, 000, 001, 10, 11, 110, 111\}$

Binary trees

$$T_n := \{binary\ trees\ having\ n\ leaves\}$$

$T_1 =$ 

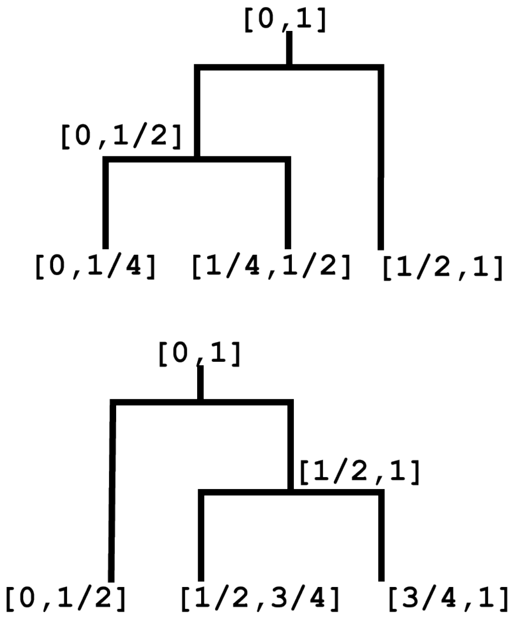
$T_2 =$ 

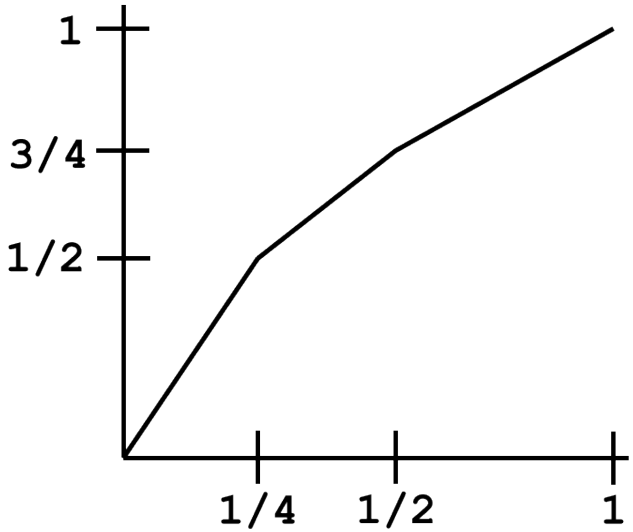
$T_3 =$ 

$T_4 =$ 

Thompson's F

We can get a map $\varphi\colon [0,1] \to [0,1]$ from a pair of binary trees.

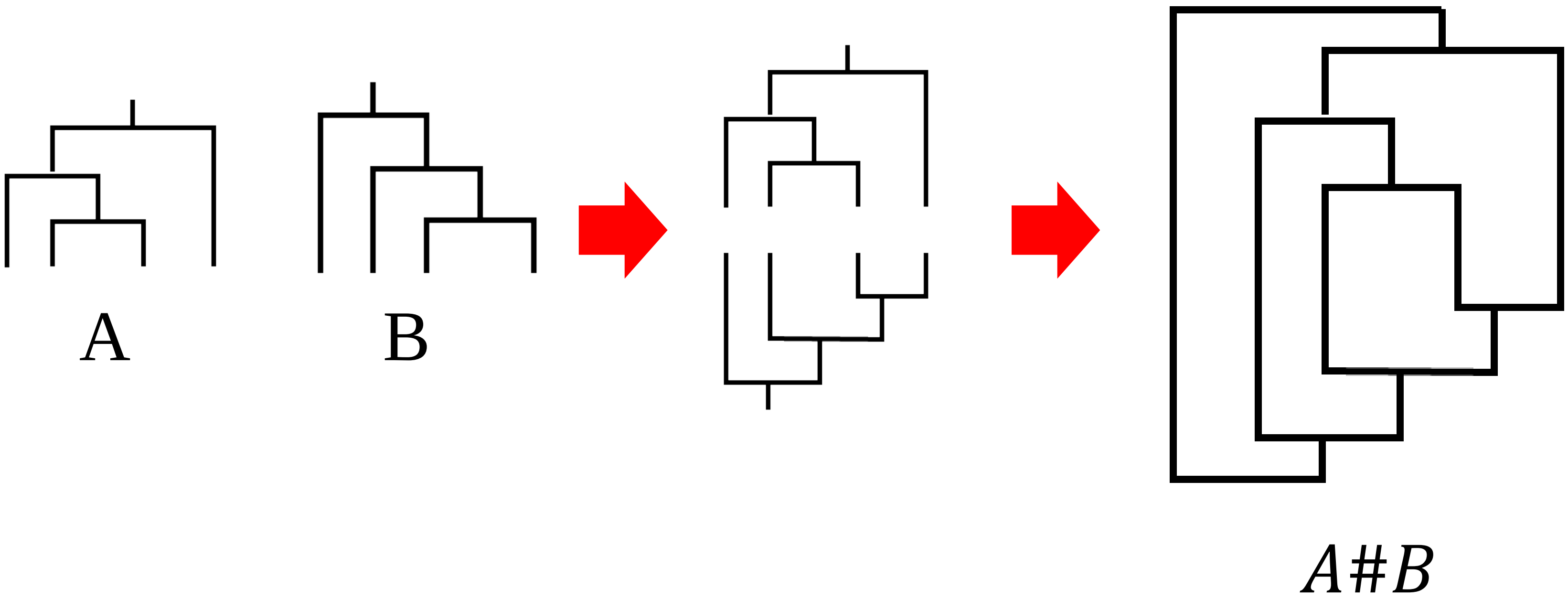




$$\varphi(x) = \begin{cases} 2x & \text{if } x \in [0, 1/4] \\ x + 1/4 & \text{if } x \in [1/4, 1/2] \\ x/2 + 1/2 & \text{if } x \in [1/2, 1] \end{cases} \in F$$

A#B

$\forall A, B \in T_n$, we get 3 regular graph when connect A and B.



Thompson's F

$\boldsymbol{Def}$(Reduced pair)
Let $A, B \in T_n$. If $A\#B$ has no C_2, then we say the pair (A, B) is ***reduced.***

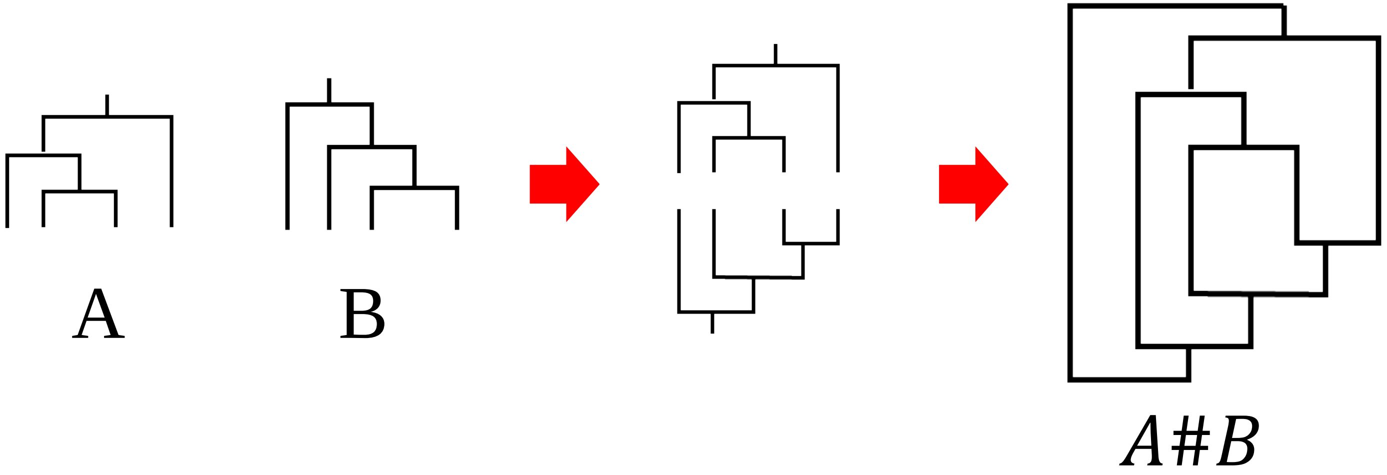
$$r(T_n^2) \coloneqq \{(A, B) \in T_n \times T_n \mid (A, B)\ \textit{is reduced}\}$$

$\boldsymbol{Theorem}$(Bowlin, Brin, 2013)

Let F be Thompson's F. There exists a bijection

$$g: F \longrightarrow \bigcup_{n \in \mathbb{N}} r(T_n^2).$$

Colorable



Def

Let $f \in F$ and $g(f) = (A, B)$.

If $A\#B$ has edge 3-coloring, we say f is **colorable.**

Rotation

Def (Rotation)
Let $u \in \{0,1\}^*$. A map $rot(u)\colon \{0,1\}^* \to \{0,1\}^*$ is defined such that

$$rot(u)(v) = \begin{cases} u0w & v = u00w \\ u10w & v = u01w \\ u11w & v = u1w \\ u & v = u0 \\ u1 & v = u \\ v & \text{otherwise} \end{cases}$$

Fact

- Let A $\in T_n$. If $w, w0 \in A, rot(w)(A) \in T_n$.
- $F \cong \langle rot(\emptyset),\ \mathrm{rot}(1) \rangle$

Weight

Lemma

$\forall n \in \mathbb{N}, \forall A, B \in T_n, \exists k$ s.t. we can change A into B with k times rotations.

Def

Let $f \in F$ and $g(f) = (A, B)$.

$w(f) \coloneqq \min\{k \mid \text{we can change } A \text{ into } B \text{ with } k \text{ times rotations}\}$

Result No.1

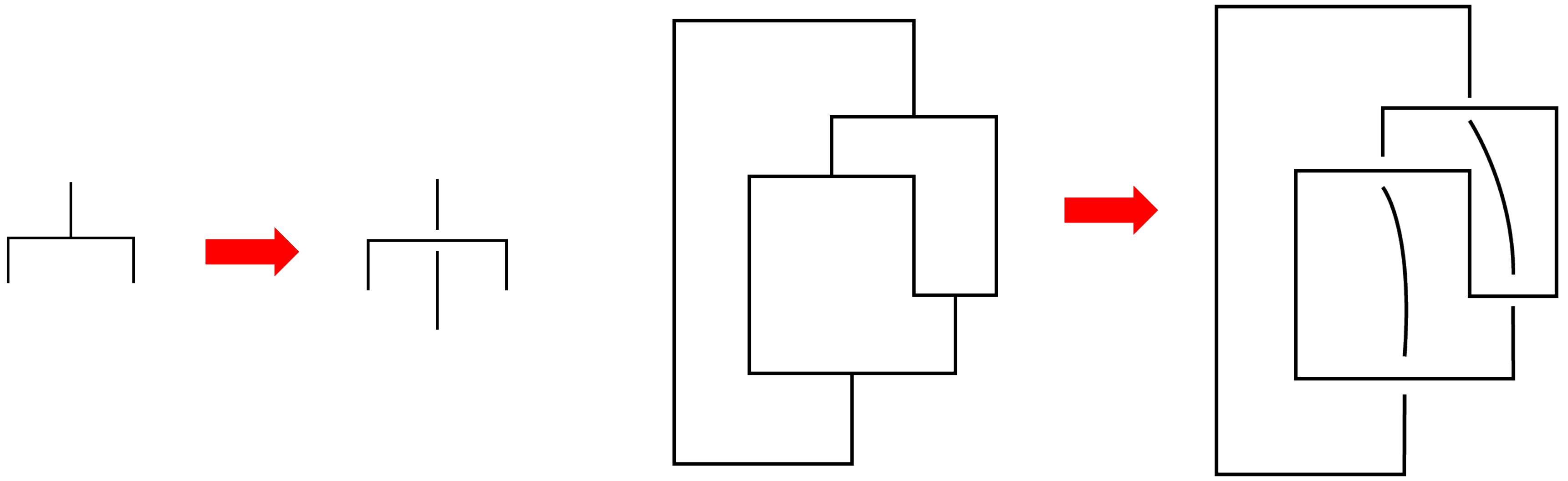
Def

$$F_n := \{f \in F \mid g(f) \in T_n \times T_n\}$$

Theorem

Four color theorem holds **if and only if**
$\forall n \in \mathbb{N}, \forall f \in F_n$ such that $\mathrm{w}(f) \geq n - 1$,
f is colorable.

It is known that we can make a link with a pair of binary trees.



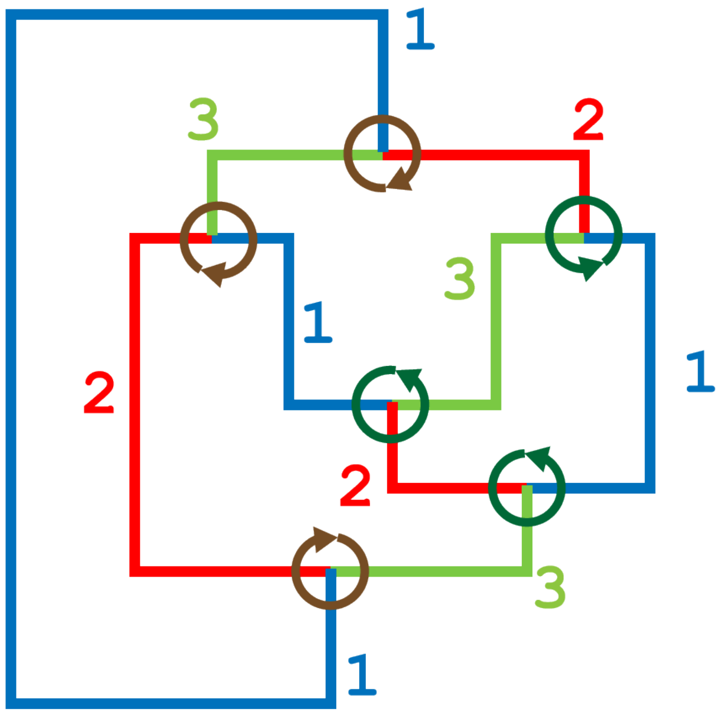
Question:

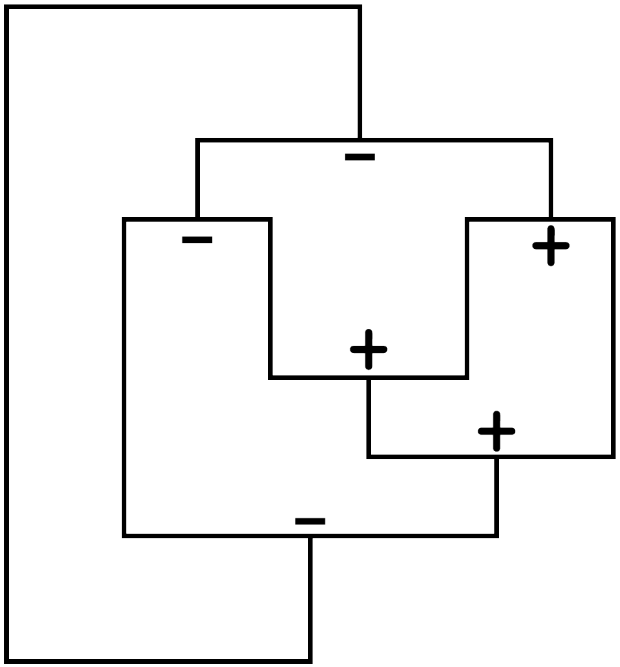
What will happen if we append information about colorings?

How to append

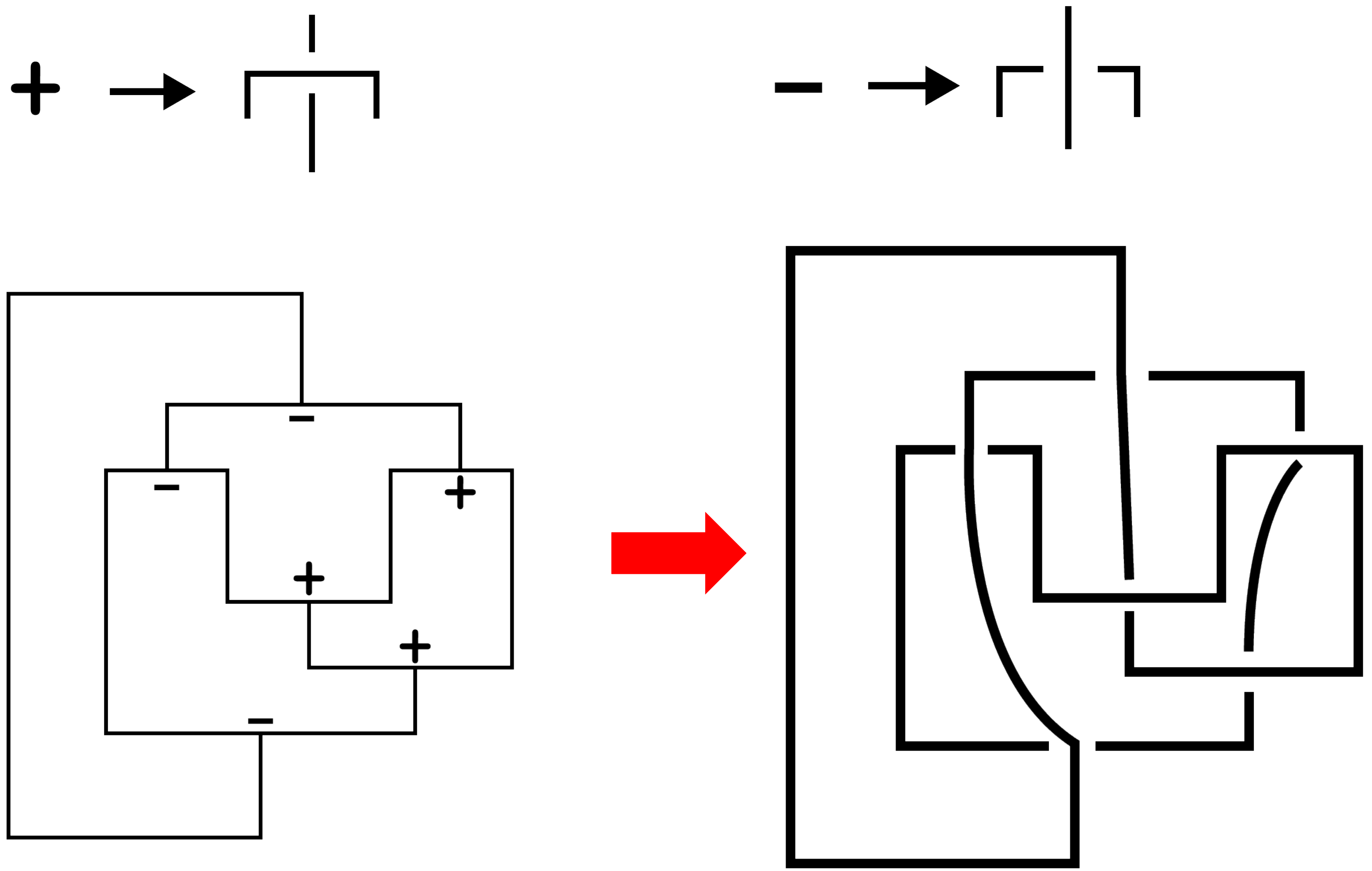
How do we append it?

We can attach + or − sign to each vertices with a coloring.





How to append



Result No.2

Def

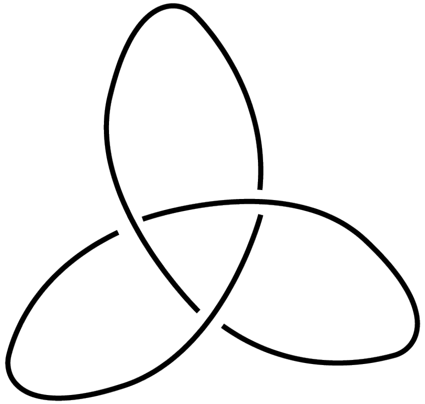
$$h: (\cup T_n \times T_n) \times \{signs\} \to \{links\}$$

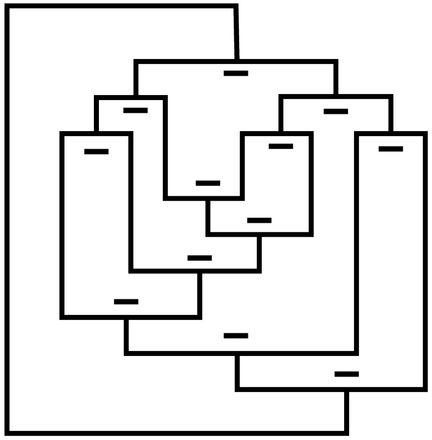
Theorem

h is surjective.

Especially, for any **knot** K, there exists $f \in F$ and a sign σ s.t

$$h(g(f), \sigma) = K.$$





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Thank you for your attention!