Maximum skew energy of tournaments

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Digraphs and skew-adjacency matrices

Definition 1

A *digraph* is a pair $D = (V, A)$ s.t. V is a finite set and $A \subset \{(x, y) \in V \times V | x \neq y\}$.

Assume $x \to y$ denotes $(x, y) \in A$.

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Definition 2

For a digraph D, the *skew-adjacency matrix* $S(D)$ is defined as

$$S(D)_{xy} = \begin{cases} 1 & \text{if } x \to y \\ -1 & \text{if } y \to x \\ 0 & \text{otherwise} \end{cases}$$

Skew energy

Definition 3

Let S be a skew-adjacency matrix and λ_i $(i = 1, \ldots, n)$ be the eigenvalues of S. Then the *skew energy*, denoted by $\varepsilon(S)$, is defined as

$$\varepsilon(S) = \sum_{i=1}^{n} |\lambda_i|.$$

The α-*skew energy*, denoted by $\varepsilon_\alpha(S)$, is defined as

$$\varepsilon_\alpha(S) = \sum_{i=1}^{n} |\lambda_i|^\alpha.$$

Note that $\varepsilon(S) = \varepsilon_1(S)$.

Tournaments

Definition 4

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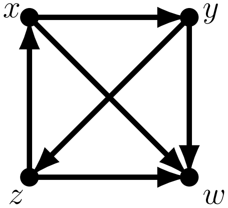
A tournament U:



The skew-adjacency matrix $S(U)$:

$$
\begin{array}{c c}
 & \begin{array}{cccc} x & y & z & w \end{array} \\
\begin{array}{c} x \\ y \\ z \\ w \end{array} & \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}
\end{array}
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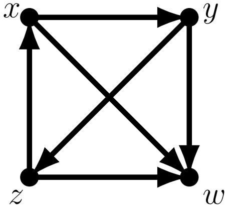

The skew-adjacency matrix $S(U)$:

$$\begin{array}{c} \\ x \\ y \\ z \\ w \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \end{array}$$

$$\phi_U(x) = x^4 + 6x^2 + 9$$

$$\mathrm{Spec}(S(U)) = \{\sqrt{-3}, \sqrt{-3}, -\sqrt{-3}, -\sqrt{-3}\}$$

$$\Rightarrow \varepsilon(S(U)) = 4\sqrt{3}$$

Properties of skew-adjacency matrices

$\mathcal{S}_n$: The set of skew-adjacency matrices of all tournaments of order n

Properties of $S \in \mathcal{S}_n$

- S is skew-symmetric and $SS^T = -S^2$.
- $\forall i = 1, \ldots, n$, $\lambda_i \in \sqrt{-1}\mathbb{R}$.
- λ is an eigenvalue of $S \Longrightarrow -\lambda$ is also an eigenvalue of S.
- n is even $\Longrightarrow$ All eigenvalues of SS^T are positive.
- $\forall i = 1, \ldots, n$, $(-S^2)_{ii} = n - 1$.

Results

Maximum	Skew energy	α-skew energy $(0 \leq \alpha \leq 2)$
$n \equiv 0 \pmod 4$	$n\sqrt{n-1}$	$n(n-1)^{\frac{\alpha}{2}}$
$n \equiv 1 \pmod 4$		
$n \equiv 2 \pmod 4$		
$n \equiv 3 \pmod 4$	$(n-1)\sqrt{n}$	

Note In skew energy, the equivalence conditions that equality holds are determined for $n \equiv 0, 3 \pmod 4$.

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$n \equiv 1 \pmod 4$		
$n \equiv 2 \pmod 4$	$2\sqrt{2n-3}$ $+(n-2)\sqrt{n-3}$	$2(2n-3)^{\frac{\alpha}{2}}$ $+(n-2)(n-3)^{\frac{\alpha}{2}}$
$n \equiv 3 \pmod 4$	$(n-1)\sqrt{n}$	

Note In skew energy, the equivalence conditions that equality holds are determined for $n \equiv 0, 3 \pmod 4$.

Comparison of α-skew energy

Lemma 5 (Efroymson-Swartz-Wendroff(1980))

Let $x_k, y_k \in \mathbb{R}_{\geq 0}$ $(k = 1, \ldots, n)$. *If* $e_i(x_1, \ldots, x_n) \leq e_i(y_1, \ldots, y_n)$ *for all* $i = 1, \ldots, n$, *then*

$$x_1^\alpha + \cdots + x_n^\alpha \leq y_1^\alpha + \cdots + y_n^\alpha \ (0 \leq \alpha \leq 1),$$

where e_i *is the elementary symmetric polynomial of degree* i.

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$$x_1^\alpha + \cdots + x_n^\alpha \leq y_1^\alpha + \cdots + y_n^\alpha \ (0 \leq \alpha \leq 1),$$

where e_i *is the elementary symmetric polynomial of degree* i.

For $A \in M_n(\mathbb{C})$, let $A[k_1, \ldots, k_i]$ be the principal submatrix of A indexed by $\{k_1, \ldots, k_i\} \subset \{1, \ldots, n\}$.

Lemma 6

Let $S, T \in \mathcal{S}_n$. *For* $\forall i = 0, 1, \ldots, n$,

$$\sum_{1 \leq k_1 < \cdots < k_i \leq n} \det SS^T[k_1, \ldots, k_i] \leq \sum_{1 \leq k_1 < \cdots < k_i \leq n} \det TT^T[k_1, \ldots, k_i]$$

$$\Longrightarrow \varepsilon_\alpha(S) \leq \varepsilon_\alpha(T) \quad (0 \leq \alpha \leq 2).$$

Maximum α-skew energy for $n \equiv 2 \pmod 4$

Theorem 7 (I.)

Let $n \equiv 2 \pmod 4$*. For* $\forall S \in \mathcal{S}_n$ *and* $0 \le \alpha \le 2$*,*

$$\varepsilon_\alpha(S) \le 2(2n-3)^{\frac{\alpha}{2}} + (n-2)(n-3)^{\frac{\alpha}{2}}.$$

If $M \in \mathcal{S}_n$ *satisfies* $MM^T = \begin{pmatrix} L(\frac{n}{2}) & 0 \\ 0 & L(\frac{n}{2}) \end{pmatrix}$*, then equality holds for* M*, where* $L(k) = (n-3)I_k + 2J_k$*.*

The spectrum of $\begin{pmatrix} L(\frac{n}{2}) & 0 \\ 0 & L(\frac{n}{2}) \end{pmatrix}$ is $\{2n-3^{(2)}, n-3^{(n-2)}\}$.

Maximum α-skew energy for $n \equiv 2 \pmod 4$

Lemma 8 (Fischer's inequality)

Let $M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ *be a positive definite symmetric matrix and* $A, C (\neq \mathbf{0})$ *be square matrices. Then* $\det M \leq \det A \det C$.
Equality holds $\Leftrightarrow B = 0$.

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Lemma 9 (Wojtas(1964))

Let $A = [a_{ij}]$ *be a positive definite symmetric matrix of order* n *with* $a_{ii} = m$ *for* $\forall i$ *and* $a = \min_{i,j} |a_{ij}|$*. Then*

$$\det A \leq \det((m-a)I_n + aJ_n).$$

Equality holds $\Leftrightarrow A \in \{D_1((m-a)I_n + aJ_n)D_2 \mid D_1, D_2 \in \mathcal{D}_n\}$,
where $\mathcal{D}_n$ *is the set of* $\{\pm 1\}$*-diagonal matrices of order* n.
Moreover $\det((m-a)I_n + aJ_n) = (m + na - a)(m-a)^{n-1}$.

Maximum α-skew energy for $n \equiv 2 \pmod 4$

Lemma 10

Let $n \equiv 2 \pmod 4$ *and* $A = [a_{ij}] = SS^T$ *for* $S \in \mathcal{S}_n$. $a_{xy} \equiv 0 \pmod 4$ *for some* (x, y) $(x \neq y)$,

$$\Rightarrow \exists P \in \mathcal{P}_n \text{ s.t. } P^T A P = \left[\begin{array}{c|c} B & C \\ \hline C^T & B' \end{array}\right],$$

where $\mathcal{P}_n$ *is the set of all permutation matrices of order* n, $B_{xy}, B'_{xy} \equiv 2 \pmod 4$ *for* $\forall x, y$ $(x \neq y)$ *and* $C_{xy} \equiv 0 \pmod 4$ *for* $\forall x, y$.

Theorem 7 (Recall)

Let $n \equiv 2 \pmod 4$. For $\forall S \in \mathcal{S}_n$ and $0 \leq \alpha \leq 2$,

$$\varepsilon_\alpha(S) \leq 2(2n-3)^{\frac{\alpha}{2}} + (n-2)(n-3)^{\frac{\alpha}{2}}.$$

If $M \in \mathcal{S}_n$ satisfies $MM^T = \begin{pmatrix} L(\frac{n}{2}) & 0 \\ 0 & L(\frac{n}{2}) \end{pmatrix}$, then equality holds, where $L(k) = (n-3)I_k + 2J_k$.

Theorem 7 (Recall)

Let $n \equiv 2 \pmod 4$. For $\forall S \in \mathcal{S}_n$ and $0 \leq \alpha \leq 2$,

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Proof. If $(SS^T)_{xy} \neq 0$ for $\forall x, y$ $(x \neq y) \Longrightarrow \min_{x,y} |(SS^T)_{xy}| = 2$.
For $\forall i = 1, \ldots, n$ and $\forall \{k_1, \ldots, k_i\} \subset \{1, \ldots, n\}$,

$$\begin{aligned} \det SS^T[k_1, \ldots, k_1] &\leq \det((n-3)I_n + 2J_n)[k_1, \ldots, k_i] \\ &= \det((n-3)I_i + 2J_i) \leq \det N[k_1, \ldots, k_i], \end{aligned}$$

where

$$N = \begin{pmatrix} L(n/2) & 0 \\ 0 & L(n/2) \end{pmatrix}.$$

If $(SS^T)_{xy} = 0$ for some x, y $(x \neq y)$, then we can assume $h \leq n/2$ and SS^T is partitioned into

$$SS^T = \begin{pmatrix} \overbrace{}^{n/2} & \overbrace{}^{n/2-h} & \overbrace{}^{h} \\ B_1 & C & D_1 \\ C^T & B_2 & D_2 \\ D_1^T & D_2^T & B_3 \end{pmatrix},$$

where
$$\begin{aligned} (B_k)_{xy} &\equiv 2 \pmod 4 \quad \text{for } \forall x, y \ (x \neq y) \ (k = 1, 2, 3), \\ C_{xy} &\equiv 2 \pmod 4 \quad \text{for } \forall x, y, \\ (D_k)_{xy} &\equiv 0 \pmod 4 \quad \text{for } \forall x, y \ (k = 1, 2) \end{aligned}.$$

Let

$$K_1 \subset \{1, \ldots, n/2\}, K_2 \subset \{n/2, \ldots, n-h\}, K_3 \subset \{n-h+1, \ldots, n\}$$

and $|K_j| = m_j$ $(j = 1, 2, 3)$ s.t. $m_1 + m_2 + m_3 = i$.

For any triad of subsets (K_1, K_2, K_3),

$$\det SS^T[K_1, K_2, K_3] = \begin{pmatrix} \overbrace{B_1[K_1]}^{m_1} & \overbrace{C'}^{m_2} & \overbrace{D_1'}^{m_3} \\ C'^T & B_2[K_2] & D_2' \\ D_1'^T & D_2'^T & B_3[K_3] \end{pmatrix}$$

$$\leq \det \begin{pmatrix} \overbrace{B_1[K_1]}^{m_1} & \overbrace{C'}^{m_2} \\ C'^T & B_2[K_2] \end{pmatrix} \det B_3[K_3]$$

$$\leq \det L(m_1 + m_2) \det L(m_3).$$

$$= \det N'[K_1, K_2, K_3],$$

where

$$N' = \begin{pmatrix} L(n-h) & 0 \\ 0 & L(h) \end{pmatrix}.$$

By direct calculation,

$$\sum \det N'[K_1, K_2, K_3] \leq \sum \det N[K_1, K_2, K_3]$$

holds.

$$\Longrightarrow \sum \det SS^T[K_1, K_2, K_3] \leq \sum \det N[K_1, K_2, K_3].$$

By $\mathrm{Spec}(N) = \{2n-3^{(2)}, n-3^{(n-2)}\}$ and Lemma 6,

$$\varepsilon_\alpha(S) \leq 2(2n-3)^{\frac{\alpha}{2}} + (n-2)(n-3)^{\frac{\alpha}{2}}.$$

□

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By $\mathrm{Spec}(N) = \{2n - 3^{(2)}, n - 3^{(n-2)}\}$ and Lemma 6,

$$\varepsilon_\alpha(S) \leq 2(2n-3)^{\frac{\alpha}{2}} + (n-2)(n-3)^{\frac{\alpha}{2}}.$$

□

Thank you for your attention.