COMPUTATIONAL COMPLEXITY OF THE VERTEX COVER PROBLEM FOR HIGHLY CONNECTED TRIANGULATIONS

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Plane (geometric) graphs: definition

Plane (geometric) graph

A triple $G = (V, E, F)$ is called a *plane* graph if:

- $V \subset \mathbb{R}^2$
- the set E consists of nondegenerate straight line segments, crossing only at their endpoints
- F denotes the set of all open (in $\mathbb{R}^2$) regions bounded by segments from E and points of V : each $f \in F$ does not intersect with any segment from E and does not contain points of V

The only unbounded set $f_\infty \in F$ is named as the *outer* face whereas bounded ones from F are *inner* faces.

Applications of plane (geometric) graphs

Spatial networks have their nodes:

- distributed in some geographical area
- communicating with each other through physical links
- can be modeled by plane graphs

Road networks

- network nodes are cities
- links denote high speed roads

PROBLEMS FOR SPATIAL NETWORKS

NETWORK MONITORING

Given a road network, locate positions for policemen at the road crossings such that each road section is monitored by some policeman.

- road crossings are modeled by points on the plane
- road sections are given in the form of straight line segments
- policemen are placed at the network nodes
- Network monitoring problem can be formulated as the Vertex Cover Problem on the respective plane graph

SPATIAL NETWORK MODELS

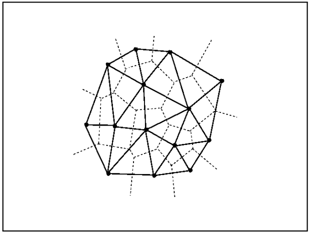
Rough but tractable models of complex network topologies are used in a form of proximity graphs:

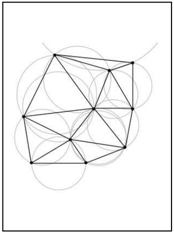
- fitting Gabriel graphs to street networks (Maniadakis 2014)
- approximating road networks with β-skeletons (Watanabe 2010)
- efficient routing algorithms for usual and special Delaunay triangulations (Bose 2001, 2012)

Examples of proximity graphs

Delaunay triangulation

Let $S \subset \mathbb{R}^2$ be a finite set of points in general position, no 4 of which are cocircular. We call a plane graph $G = (V, E, F)$ by a *Delaunay triangulation* if $V = S$, $e = [u, v] \in E$ iff there is a disk $d(u, v)$ such that $u, v \in \mathrm{bd}\, d(u, v)$ and $S \cap \mathit{int}\, d(u, v) = \varnothing$.





Basic problem

Vertex Cover problem (VC)

Given a simple graph $G = (V, E)$ find the smallest cardinality subset $V' \subseteq V$ such that $V' \cap e \neq \varnothing$ for every $e = \{u, v\} \in E$.

Our questions and motivation

- what is the **VC** problem computational complexity in the case where G belongs to the class of usual and special Delaunay triangulations?
- as we mentioned above, usual and special Delaunay triangulations are convenient models of network topologies
- thus, studying the **VC** problem complexity is of interest for the class of DT!

Basic problem

Vertex Cover problem (VC)

Given a simple graph $G = (V, E)$ find the smallest cardinality subset $V' \subseteq V$ such that $V' \cap e \neq \varnothing$ for every $e = \{u, v\} \in E$.

Combining results [Bodlaender, 1998] and [Dillencourt, 1990]:

Related work

VC is polynomially solvable in the class of outerplane Delaunay triangulations of the form $G = (V, E)$ such that every $v \in V$ is also a vertex of $\mathrm{conv}\, V$.

SPECIAL DELAUNAY TRIANGULATIONS

Let $\nabla(p,\lambda) = p + \lambda\nabla = \{x \in \mathbb{R}^2 : x = p + \lambda a,\ a \in \nabla\}$ for some $p \in \mathbb{R}^2$ and $\lambda > 0$ where ∇ is the unit sided oriented equilateral triangle whose barycenter is the origin and one of its vertices is on the negative y-axis.

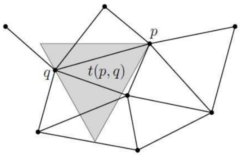
TD-DELAUNAY TRIANGULATION

Let $S \subset \mathbb{R}^2$ be a set of points, straight line through any pair u and v from S makes neither of angles $0^\circ, 60^\circ, 120^\circ$ with horizontal. A plane graph $G = (V, E, F)$ with $V = S$ is called a *TD-Delaunay triangulation* iff $e = [u, v] \in E$ whenever there exists $p \in \mathbb{R}^2$ and $\lambda > 0$ such that $u, v \in \operatorname{bd} \nabla(p,\lambda)$ and $S \cap \operatorname{int} \nabla(p,\lambda) = \varnothing$.

Special Delaunay triangulations

TD-Delaunay triangulation

Let $S \subset \mathbb{R}^2$ be a set of points. A plane graph $G = (V, E, F)$ with $V = S$ is called a *TD-Delaunay triangulation* iff $e = [u, v] \in E$ whenever there exists $p \in \mathbb{R}^2$ and $\lambda > 0$ such that $u, v \in \mathrm{bd}\, \nabla(p, \lambda)$ and $S \cap \mathrm{int}\, \nabla(p, \lambda) = \varnothing$.



Hardness for TD-Delaunay triangulations

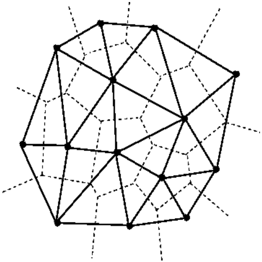
Our main geometric result

The **VC** problem is strongly NP-hard within the class of 4-connected TD-Delaunay triangulations whose degree is of the order $O(\log n)$, where n is the number of triangulation vertices.

Some definitions

Planar triangulation

A planar graph is called a planar triangulation if it admits an embedding in the form of a plane graph $G = (V, E, F)$ whose faces are triangles except for possibly the outer one.



Hardness proof sketch

First, we get the following result.

Our main combinatorial result

The **VC** problem is strongly NP-hard within the class of 4-connected planar triangulations with triangular outer face whose degree is of the order $O(\log n)$.

Next, we apply the result from graph drawing.

Bonichon, 2010

Every planar triangulation with triangular outer face can be embedded on the plane in the form of TD-Delaunay triangulation in polynomial time and space.

HARDNESS PROOF SKETCH

OUR MAIN COMBINATORIAL RESULT

The **VC** problem is strongly NP-hard in the class of 4-connected planar triangulations with triangular outer face whose degree is of the order $O(\log n)$.

DILLENCOURT, 1996

Every 4-connected planar triangulation with triangular outer face can be embedded on the plane in the form of Delaunay triangulation (possibly not in polynomial time and space).

CONJECTURE

The **VC** problem is strongly NP-hard within the class of 4-connected Delaunay triangulations.

Our hardness proof technique and related ones

Mohar, 2001

Maximum Independent Set problem (**MIS**) is strongly NP-hard in the class of 2-connected 3-regular planar graphs.

Independent Set

Given a graph $G = (V, E)$, a set $V' \subseteq V$ is called independent if no two vertices of V' are adjacent.

Our hardness proof technique and related ones

Mohar, 2001

Maximum Independent Set problem (**MIS**) is strongly NP-hard in the class of 2-connected 3-regular planar graphs.

Da Lozzo, Rutter, 2016 (unpublished)

MIS and **VC** are strongly NP-hard in the class of 3-connected planar triangulations with triangular outer face.

Their construction starts from 2-connected 3-regular planar graph by sequentially replacing nontriangular faces with special triangulated gadgets.

Our hardness proof technique and related ones

Mohar, 2001, dual form

Maximum Facial Independent Set problem (**Facial MIS**) is strongly NP-hard in the class of planar triangulations (possibly containing parallel edges).

Facial Independent Set

Given a plane embedding $G = (V, E, F)$, a set $F' \subseteq F$ of faces is independent if no pair $f_1, f_2 \in F'$ shares an edge.

Our hardness technique and related ones

Mohar, 2001, dual form

Maximum Facial Independent Set problem (**Facial MIS**) is strongly NP-hard in the class of planar triangulations (possibly containing parallel edges).

Comparison of our technique with related ones

- our (nonseq.) construction uses dual form of Mohar result
- it leads to the hardness result in the narrower class of 4-connected triangulations (isomorphic to Delaunay triangulations) in contrast to Da Lozzo's construction which creates triangulations having non-facial 3-cycles
- it gives $O(\log n)$ bound on the triangulation degree

CONCLUSION

SUMMARY

- hardness of the **VC** problem is proved for 4-connected planar and TD-Delaunay triangulations
- hardness of **VC** for Delaunay triangulations is likely to hold

High connectivity and low complexity of VC?

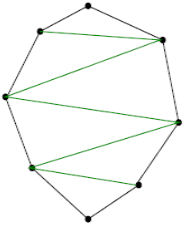
Question

- **VC** has high complexity for highly connected planar triangulations
- what is the connectivity of triangulations from the graph classes for which **VC** is polynomially solvable ?

Connectivity of outerplanar triangulations

Outerplanar triangulation

A 2-connected planar triangulation is called outerplanar if it admits an embedding in the form of a plane graph $G = (V, E, F)$ where the set V is contained on the boundary of its outer face $f_\infty \in F$.



Connectivity of outerplanar triangulations

Outerplanar triangulation

A 2-connected planar triangulation is called outerplanar if it admits an embedding in the form of a plane graph $G = (V, E, F)$ where the set V is contained on the boundary of its outer face $f_\infty \in F$.

Bodlaender, 1998

The **VC** problem is polynomially solvable in linear time in the class of outerplanar triangulations.

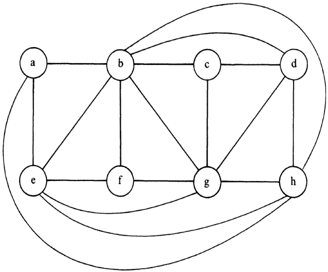
Connectivity of outerplanar triangulations

Outerplanar triangulations can not be 4-connected.

CONNECTIVITY OF CHORDAL TRIANGULATIONS

CHORDAL TRIANGULATION

A 3-connected planar triangulation is called chordal if each its cycle of length more than 3 has a chord, i.e., a graph edge which connects two non-consecutive vertices of that cycle.



CONNECTIVITY OF CHORDAL TRIANGULATIONS

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A 3-connected planar triangulation is called chordal if each its cycle of length more than 3 has a chord, i.e., a graph edge which connects two non-consecutive vertices of that cycle.

GAVRIL, 1972

The **VC** problem is polynomially solvable in the class of chordal triangulations.

CONNECTIVITY OF CHORDAL TRIANGULATIONS

Chordal triangulations can not be 4-connected for $n > 4$.