Strongly regular graphs with the same parameters as the symplectic graph

Sho Kubota

Graduate School of Information Sciences
Tohoku University

August 20, 2016
Graphs and Groups, Spectra and Symmetries
Novosibirsk State University

Equitable partitions

X : a graph
$\pi = \{C_1, \ldots, C_t\}$: a partition of $V(X)$
π is called an equitable partition if $\forall i, j \in [t]$, $\forall x, x' \in C_i$,

$$|N(x) \cap C_j| = |N(x') \cap C_j|.$$

Equitable partitions

X : a graph
$\pi = \{C_1, \ldots, C_t\}$: a partition of $V(X)$
π is called an equitable partition if $\forall i, j \in [t]$, $\forall x, x' \in C_i$,

$$|N(x) \cap C_j| = |N(x') \cap C_j|.$$

Example

G : a subgroup of $\mathrm{Aut}(X)$
π : the orbit partition of G

$$\Longrightarrow \quad \pi \text{ is an equitable partition.}$$

Godsil-McKay switching

Theorem 1 (Godsil-McKay, 1982)

X : a graph
$\pi = \{C_1, \ldots, C_t, D\}$: a partition of $V(X)$
Assume that π satisfies

- $\{C_1, \ldots, C_t\}$ is an equitable partition of $V(X) \setminus D$,
- $\forall x \in D$, $\forall i \in [t]$, $|N(x) \cap C_i| = 0, \frac{1}{2}|C_i|$ or $|C_i|$.

Construct a new graph X' by interchanging adjacency and nonadjacency between $x \in D$ and the vertices in C_i whenever x has $\frac{1}{2}|C_i|$ neighbors in C_i.

$$\Longrightarrow \quad \mathrm{Spec}(X) = \mathrm{Spec}(X')$$

We will call this special cell D a GM cell.

Outline of this study

X : a SRG
X' : a graph obtained from X by an operation s.t.

$$\mathrm{Spec}(X') = \mathrm{Spec}(X) \text{ (and } X' \not\simeq X\text{)}$$

Outline of this study

X : a SRG
X' : a graph obtained from X by an operation s.t.

$$\mathrm{Spec}(X') = \mathrm{Spec}(X) \text{ (and } X' \not\simeq X)$$
$$\Downarrow$$

X' is also a SRG with the same parameters as X.

Outline of this study

X : a SRG
X' : a graph obtained from X by an operation s.t.

$$\mathrm{Spec}(X') = \mathrm{Spec}(X) \text{ (and } X' \not\simeq X)$$
$$\Downarrow$$

X' is also a SRG with the same parameters as X.
"an operation" = GM switching

Outline of this study

X : a SRG
X' : a graph obtained from X by an operation s.t.

$$\mathrm{Spec}(X') = \mathrm{Spec}(X) \text{ (and } X' \not\simeq X)$$
$$\Downarrow$$

X' is also a SRG with the same parameters as X.
"an operation" $=$ GM switching w.r.t. an orbit partition

Outline of this study

X : a SRG
X' : a graph obtained from X by an operation s.t.

$$\mathrm{Spec}(X') = \mathrm{Spec}(X) \text{ (and } X' \not\simeq X)$$
$$\Downarrow$$

X' is also a SRG with the same parameters as X.
"an operation" = GM switching w.r.t. an orbit partition

What SRG do we consider? $\rightarrow$ The symplectic graph.

The symplectic graphs

Let $R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The symplectic graph $Sp(2\nu, 2)$ over $\mathbb{F}_2$ is the graph defined by the following:

$$V(Sp(2\nu, 2)) = \mathbb{F}_2^{2\nu} \setminus \{\mathbf{0}\},$$
$$E(Sp(2\nu, 2)) = \{xy \mid x^T K y = 1\},$$

where $K = I_\nu \otimes R$.

Proposition 2

The symplectic graph $Sp(2\nu, 2)$ is a SRG with parameters $(2^{2\nu} - 1,\ 2^{2\nu-1},\ 2^{2\nu-2},\ 2^{2\nu-2})$

The automorphism group of $Sp(2\nu, 2)$

Theorem 3 (Tang and Wan, 2006)

$$\mathrm{Aut}(Sp(2\nu, 2)) \simeq \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T K A = K\}.$$

But $Sp(2\nu, 2)$ is vertex-transitive.

The automorphism group of $Sp(2\nu, 2)$

Theorem 3 (Tang and Wan, 2006)

$$\mathrm{Aut}(Sp(2\nu, 2)) \simeq \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T K A = K\}.$$

But $Sp(2\nu, 2)$ is vertex-transitive.

We consider GM switching w.r.t. an orbit partition.

The automorphism group of $Sp(2\nu, 2)$

Theorem 3 (Tang and Wan, 2006)

$$\mathrm{Aut}(Sp(2\nu, 2)) \simeq \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T K A = K\}.$$

But $Sp(2\nu, 2)$ is vertex-transitive.

We consider GM switching w.r.t. an orbit partition.

$$\downarrow$$

We must choose proper subgroups of $\mathrm{Aut}(Sp(2\nu, 2))$.

The automorphism group of $Sp(2\nu, 2)$

Theorem 3 (Tang and Wan, 2006)

$$\mathrm{Aut}(Sp(2\nu, 2)) \simeq \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T KA = K\}.$$

But $Sp(2\nu, 2)$ is vertex-transitive.

We consider GM switching w.r.t. an orbit partition.

$$\downarrow$$

We must choose proper subgroups of $\mathrm{Aut}(Sp(2\nu, 2))$.

$$\downarrow$$

$$\mathrm{Aut}(Sp(2\nu, 2))_{\mathcal{E}} \quad \mathrm{Aut}(Sp(2\nu, 2))_S$$

The automorphism group of $Sp(2\nu, 2)$

Theorem 3 (Tang and Wan, 2006)

$$\mathrm{Aut}(Sp(2\nu, 2)) \simeq \{A \in GL_{2\nu}(\mathbb{F}_2) \mid A^T K A = K\}.$$

But $Sp(2\nu, 2)$ is vertex-transitive.

We consider GM switching w.r.t. an orbit partition.

$$\downarrow$$

We must choose proper subgroups of $\mathrm{Aut}(Sp(2\nu, 2))$.

$$\downarrow$$

$$\mathrm{Aut}(Sp(2\nu, 2))_{\mathcal{E}} \quad \mathrm{Aut}(Sp(2\nu, 2))_S$$

Fixing a special 4-subset

$X = Sp(2\nu, 2)$

v_1, v_2, v_3 : three distinct vertices of $V(X)$ s.t.

- They are linearly independent
- $v_i^T K v_j = 0$ ($\forall i, j \in [3]$)

$S = \{v_1, v_2, v_3, v_4\}$, where $v_4 = v_1 + v_2 + v_3$.
We consider the action of $\mathrm{Aut}(X)_S$.

What we should do

- Determination of the orbit partition of $\mathrm{Aut}(X)_S$
- Finding GM cells

Why do we consider S ?

Abiad and Haemers considered the following partition.

$$\{S, V(X) \setminus S\}$$

Why do we consider S ?

Abiad and Haemers considered the following partition.

$$\{S, V(X) \setminus S\}$$
$$\uparrow$$
GM cell

They obtained many SRGs with the same parameters as $Sp(2\nu, 2)$.

The orbit partition of $\mathrm{Aut}(X)_S$

$x \in V(X)$.
Since $x^T K v_1 + x^T K v_2 + x^T K v_3 + x^T K v_4 = x^T K \mathbf{0} = 0$,

$$\#\{i \in [4] \mid x^T K v_i = 1\} = 0, 2, 4.$$

The orbit partition of $\mathrm{Aut}(X)_S$

$x \in V(X)$.
Since $x^T K v_1 + x^T K v_2 + x^T K v_3 + x^T K v_4 = x^T K \mathbf{0} = 0$,

$$\#\{i \in [4] \mid x^T K v_i = 1\} = 0, 2, 4.$$

Thus,

$$V(X) = S_0 \sqcup S_2 \sqcup S_4,$$

where $S_i = \{x \in V(X) \mid \#\{j \in [4] \mid x^T K v_j = 1\} = i\}$.

The orbit partition of $\mathrm{Aut}(X)_S$

$x \in V(X)$.
Since $x^T K v_1 + x^T K v_2 + x^T K v_3 + x^T K v_4 = x^T K \mathbf{0} = 0$,

$$\#\{i \in [4] \mid x^T K v_i = 1\} = 0, 2, 4.$$

Thus,

$$V(X) = S_0 \sqcup S_2 \sqcup S_4,$$

where $S_i = \{x \in V(X) \mid \#\{j \in [4] \mid x^T K v_j = 1\} = i\}$.
Note that S, $\langle S \rangle \subset S_0$ and $\langle S \rangle^g = \langle S \rangle$ for $g \in \mathrm{Aut}(X)_S$.

Proposition 4

The orbit partition of $V(X)$ of $\mathrm{Aut}(X)_S$ is

$$\{S, T, S_0 \setminus (S \cup T), S_2, S_4\},$$

where $T = \langle S \rangle \setminus (S \cup \{\mathbf{0}\}) = \{v_1 + v_2, v_2 + v_3, v_3 + v_1\}$.

Finding GM cells

$$\{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$$

Finding GM cells

$$\{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$$

Finding GM cells

$$\{S, T, S_0 \setminus (S \cup T), S_2, S_4\}$$

We obtain three switched graphs $X^S, X^{S_0 \setminus (S \cup T)}, X^{S_4}$.
Actually,

$X^S \simeq$ switched $Sp(2\nu, 2)$ by Abiad and Haemers.

Not being isomorphic

Four graphs $X, X^{S}, X^{S_0 \setminus (S \cup T)}, X^{S_4}$ are not isomorphic ?

Not being isomorphic

Four graphs $X, X^S, X^{S_0 \setminus (S \cup T)}, X^{S_4}$ are not isomorphic ?

$\rightarrow$ We consider the number of common neighbors of three vertices.

Not being isomorphic

Four graphs $X, X^S, X^{S_0 \setminus (S \cup T)}, X^{S_4}$ are not isomorphic ?

$\rightarrow$ We consider the number of common neighbors of three vertices.

X : a graph
For $x, y, z \in V(X)$, define

$$\mathcal{N}_X[xy|z] = \left\{ w \in V(X) \setminus \{x, y, z\} \;\middle|\; \begin{array}{l} w \sim x, \\ w \sim y, \\ w \not\sim z \end{array} \right\}.$$

Not being isomorphic

Four graphs $X, X^S, X^{S_0 \setminus (S \cup T)}, X^{S_4}$ are not isomorphic ?

$\rightarrow$ We consider the number of common neighbors of three vertices.

X : a graph
For $x, y, z \in V(X)$, define

$$\mathcal{N}_X[xy|z] = \left\{ w \in V(X) \setminus \{x, y, z\} \;\middle|\; \begin{array}{l} w \sim x, \\ w \sim y, \\ w \not\sim z \end{array} \right\}.$$

$|\mathcal{N}_X[xyz|\,]| = \#$ common neighbors of three vertices in X

X : a graph
$\pi = \{C_1, \ldots, C_t, C_{t+1}\}$: an orbit partition
Assume that π has a GM cell $D = C_{t+1}$.

X : a graph
$\pi = \{C_1, \ldots, C_t, C_{t+1}\}$: an orbit partition
Assume that π has a GM cell $D = C_{t+1}$.
For any $i \in [t]$,

$$\{|N(x) \cap C_i| \mid x \in D\} = \{0\}, \left\{\frac{1}{2}|C_i|\right\} \text{ or } \{|C_i|\},$$

so

$$[t] = \mathcal{C}_0 \sqcup \mathcal{C}_{\frac{1}{2}} \sqcup \mathcal{C}_1,$$

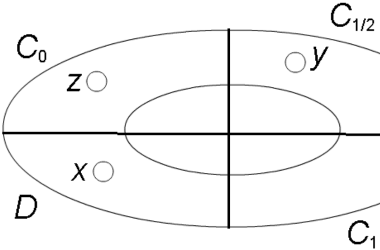
where $\mathcal{C}_j = \left\{ i \in [t] \,\middle|\, |N(x) \cap C_i| = j|C_i| \, (\forall x \in D) \right\}$.

X' : the switched graph
x, y, z : three distinct vertices of $V(X)$
The set of pairs of vertices involved with switching is

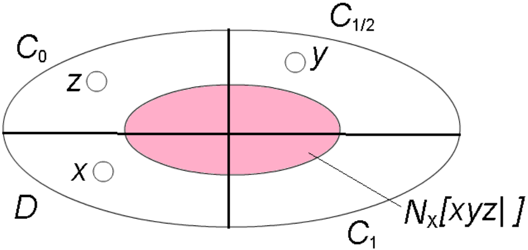
$$\bigsqcup_{i \in \mathcal{C}_{\frac{1}{2}}} \left\{ \{v, w\} \,\middle|\, v \in D, w \in C_i \right\}.$$

Considering above, we have to consider many cases to find $|\mathcal{N}_{X'}[xyz|\,]|$, but in this talk, we introduce a special case.

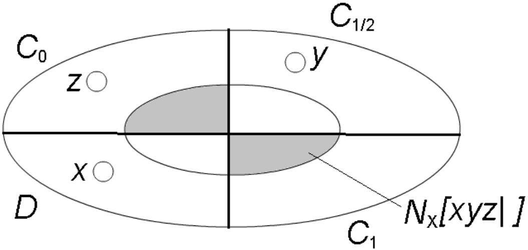
Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



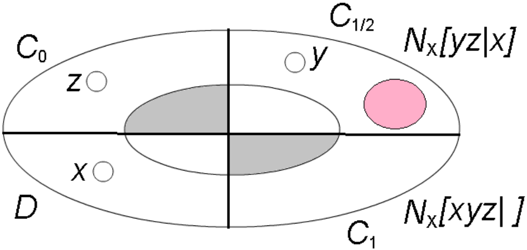
Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



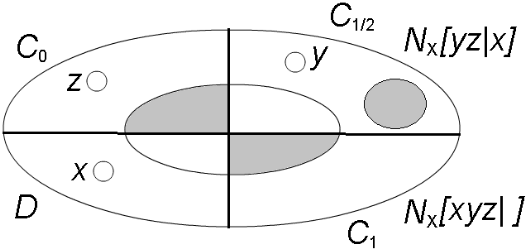
Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



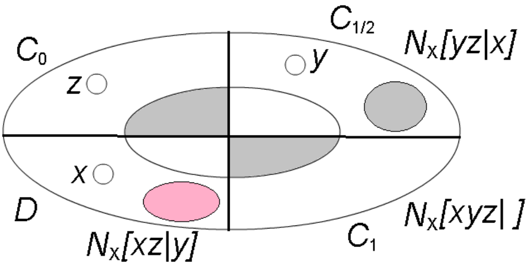
Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



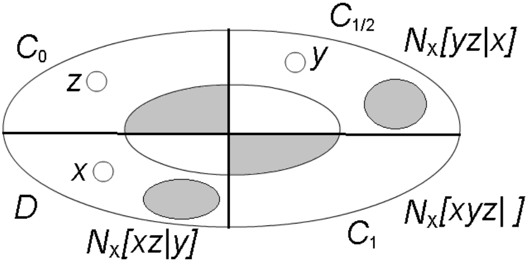
Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



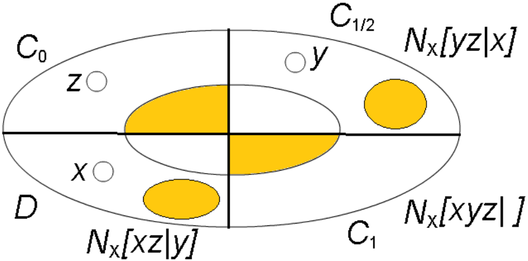
Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



Assume that $x \in D = C_{t+1}$, $y \in C_k$ and $z \in C_l$, where $k \in \mathcal{C}_{\frac{1}{2}}$ and $l \in \mathcal{C}_0 \cup \mathcal{C}_1$.



Therefore, $|\mathcal{N}_{X'}[xyz|\,]|$ is equal to

$$\sum_{i \in \mathcal{C}_0 \sqcup \mathcal{C}_1} |C_i \cap \mathcal{N}_X[xyz|\,]| + \sum_{i \in \mathcal{C}_{\frac{1}{2}}} |C_i \cap \mathcal{N}_X[yz|x]| + |D \cap \mathcal{N}_X[xz|y]|$$

The non-zero minimum number

We investigate the non-zero minimum number of common neighbors of three distinct vertices.
As a result,

The non-zero minimum number

We investigate the non-zero minimum number of common neighbors of three distinct vertices.
As a result,

X	X^S	$X^{S_0 \setminus (S \cup T)}$	X^{S_4}
$2^{2\nu-3}$	1	$2^{2\nu-5} - 2$	$2^{2\nu-5}$

The four graphs are not isomorphic to each other.

The non-zero minimum number

We investigate the non-zero minimum number of common neighbors of three distinct vertices.
As a result,

X	X^S	$X^{S_0 \setminus (S \cup T)}$	X^{S_4}
$2^{2\nu-3}$	1	$2^{2\nu-5}-2$	$2^{2\nu-5}$

The four graphs are not isomorphic to each other.

Thank you for attention !!