A family of regular coherent non-Schurian graphs, related to extremal graph theory

Matan Ziv-Av
(Joint work with Mikhail Klin and Guillermo Pineda-Villavicencio)

Ben-Gurion University of the Negev

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Computer tools

- In this project we use computer tools extensively.
- The main computer system in use is GAP.
- GAP packages: GRAPE, DESIGN, FinInG.
- The GAP package COCO-II is a work in progress.
- External tools with some interfacing to GAP: COCO, stabil, bliss.
- Database of known cages: web page of Gordon Royle.

Color graphs

- Let Ω be a finite set.
- A color graph with vertex set Ω is a coloring of the arcs of the complete digraph on Ω (with loops).
- In other words, $(\Omega, \mathcal{R})$ is a color graph if $\mathcal{R} = \{R_0, \ldots, R_d\}$ is a partition of $\Omega \times \Omega$.
- The order of the color graph is $|\Omega|$.
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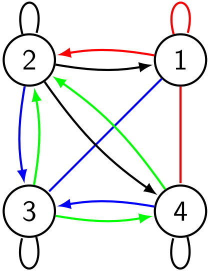

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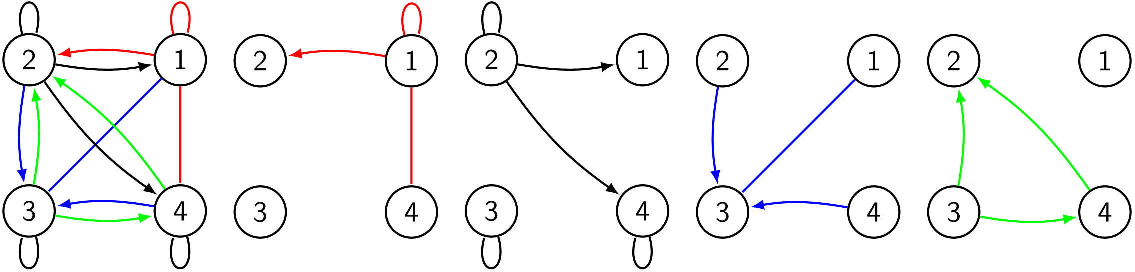

$\mathfrak{c}$ Γ_1 Γ_2 Γ_3 Γ_4

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  3. For every $R_i, R_j, R_k \in \mathcal{R}$ and for all $(x, y) \in R_k$, the number of z such that $(x, z) \in R_i$ and $(z, y) \in R_j$ does not depend on x, y.
- The relations $R_0, \ldots, R_d$ are called the basic relations of $\mathfrak{M}$. The corresponding digraphs (Ω, R_i) are called the basic graphs.
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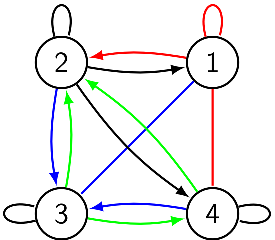

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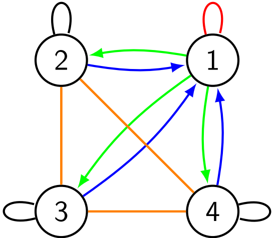

Coherent algebras

- A matrix subalgebra $\mathcal{A}$ of $\mathbb{C}^{n\times n}$ is called a coherent algebra if $\mathcal{A}$ contains I_n, $\mathcal{J}_n$ (the all-1 matrix), and is closed under transposition and under Schur-Hadamard product (entry-wise product).
- A coherent algebra has a basis $B = \{A_0, \ldots, A_d\}$ of $0, 1$ matrices, called the first standard basis of $\mathcal{A}$.
- The matrices $A_0, \ldots, A_d$ are the adjacency matrices of graphs $\Gamma_0, \ldots, \Gamma_d$ which are the basic graphs of a coherent configuration.
- If I_n is a member of the first standard basis, the coherent algebra is called homogeneous.
- If $\mathcal{C}$ is a coherent algebra which is a subalgebra of a coherent algebra $\mathcal{A}$, we say that $\mathcal{C}$ is a coherent subalgebra of $\mathcal{A}$.
- Corresponding combinatorial language: merging (or fusion) of a coherent configuration, merging some relations together.

Schurian coherent configurations

- A source for coherent configurations is permutation groups.
- If $G \leq Sym(\Omega)$ is a permutation group, an orbit of G of $\Omega \times \Omega$ is called a 2-orbit of G.
- The set $2 - Orb(G, \Omega)$ of all 2-orbits of G is a coherent configuration.
- In algebraic language, this is the centralizer algebra of G.
- If G is transitive, we get an association scheme.
- A coherent configuration which can be represented as the 2-orbits of a permutation group is called Schurian.
- Not all coherent configurations are Schurian (the smallest example for non-Schurian coherent configuration has order 14).

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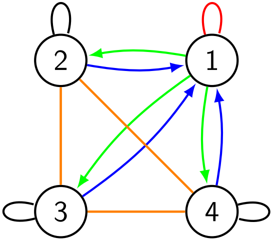

$2 - Orb(\langle(2,3),(2,3,4)\rangle, \{1,2,3,4\})$

Weisfeiler-Leman algorithm

- Every $n \times n$ matrix A, belongs to a smallest (rank) coherent algebra.
- This algebra $\ll A \gg$ is called the coherent closure of A.
- The Weisfeiler-Leman algorithm calculates the coherent closure in polynomial time.
- In short: replace distinct entries of A with distinct non-commuting variables, and calculate A^2. Repeat until number of distinct entries does not increase.

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 1 & 4 & 0 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_2 \\ x_1 & x_3 & x_0 \end{pmatrix} \to \begin{pmatrix} x_0^2 + x_1x_2 + x_2x_1 & x_0x_1 + x_1x_0 + x_2x_3 & \cdots \\ & \ddots & \\ & & \end{pmatrix} \to \cdots$$

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- The coherent closure of a graph Γ is a merging of $2 - Orb(Aut(\Gamma))$.

Coherent graph

- The arc set of every graph is a union of relations of its coherent closure.
- If the whole graph is a basic graph of its coherent closure, we call it a coherent graph.
- A (simple) coherent graph (with no isolated vertices) is regular.
- A strongly regular graph is coherent (in fact, the graph and its complement are the classes of a rank 3 association scheme).
- Thus the coherency property is between regularity and strong regularity.
- For comparison: a vertex transitive graph is regular, an arc-transitive graph is coherent, a rank 3 graph is strongly regular.

Extremal graph theory

- Extremal graph theory (EGT) studies graphs which are extreme with respect to some prescribed properties.
- For example: (Turán) Maximal number of edges in a graph of order n not with no k-clique is $\frac{n^2(k-2)}{2(k-1)}$.
- EGT is connected with AGT by the fact that in many cases, extremal graphs are highly symmetric.

Cages

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- For $g = 3$, a (k, g)-cage is K_{k+1}, so $n(k, 3) = k + 1$.
- For $g = 4$, a (k, g)-cage is the complete bipartite graph $K_{k,k}$, so $n(k, 4) = 2k$.
- For $k = 2$, a (k, g)-cage is a g-cycle, so $n(2, g) = g$.

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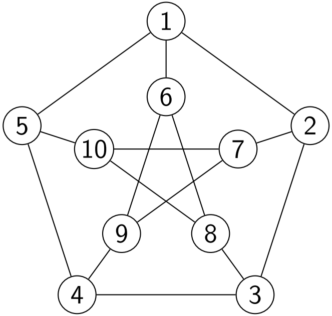
- For $k = 2$, a (k, g)-cage is a g-cycle, so $n(2, g) = g$.
- The problem is more interesting for higher k's and g's.
- A regular graph of valency k and girth g always exists. Furthermore, $n(k, g) \leq 2kq^{\frac{3g-a}{4}}$, where q is the smallest odd prime power such that $k \leq q$, and $a = 16, 11, 14, 13$ for $g \cong 0, 1, 2, 3 \pmod 4$.

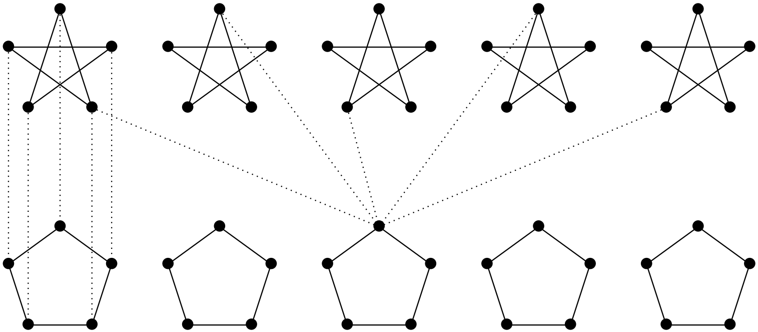
Moore bound

- Moore bound: for odd g, $n(k,g) \geq 1 + k + k(k-1) + \cdots + k(k-1)^{\frac{g-3}{2}}$.
  For even g: $n(k,g) \geq 2 + (k-1) + (k-1)^2 + \cdots + (c-1)^{\frac{g-2}{2}}$.
- Graphs which attain this lower bound are called Moore graphs.
- For $g = 5$, $n(k,5) \geq k^2 + 1$.
- Moore graphs of girth 5 are possible only for $k \in \{2, 3, 7, 57\}$.
- For $k = 3$, the unique $(3,5)$-cage is the Petersen graph.



Moore bound

- For $k = 7$, the unique $(7, 5)$-cage is the Hoffman-Singleton graph.



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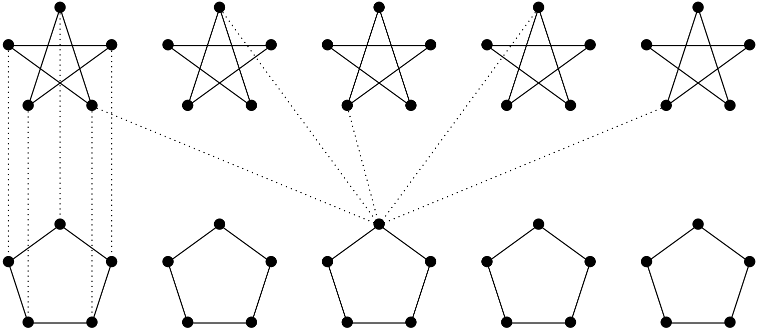

- This is the Robertson model.
- For $k = 57$ the existence of the Moore graph is unknown.

Geometric cages, $g = 6, 8, 12$

- The incidence (Levi) graph of a projective plane of order q is a $(q+1, 6)$-cage which attains the Moore bound.
- For example, the Heawood graph is the unique $(3, 6)$-cage.
- Conversely, a $(q+1, 6)$-cage which attains the Moore bound is the Levi graph of a projective plane.

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- Conversely, a $(q+1, 6)$-cage which attains the Moore bound is the Levi graph of a projective plane.
- The Levi graph of a generalized quadrangle of order q, $GQ(q)$ is a $(q+1, 8)$-cage which attains the Moore bound.
- The Levi graph of a generalized hexagon of order q is a $(q+1, 12)$-cage which attains the Moore bound.

Status of search for cages

Higher valency cages - Mozilla Firefox

http://staffhome.ecm.uwa.edu.au/~00013890/remote/cages/allcages.html

k\g	5	6	7	8	9	10	11	12	13	14	15
3	=10	=14	=24	=30	=58	=70	=112	=126	272	384	620
4	=19	=26	67 [Exoo]	=80	275[Exoo]	384[Exoo]		=728			
5	=30	=42	152[McK/Yan]	=170				=2730			
6	=40	=62	294[McK/Yan]	=312				=7812			
7	=50	=90									
8	80[Roy01]	=114		=800				=39216			
9	98[Mur 79]	=146		=1170				=74898			
10	126[Exo 01b]	=182		=1640				=132860			
11	160[Exo 01b]	240[Won 86]									
12	203[Exoo]	=266		=2928				=354312			
13	240[Exo 01b]										
14	312[Exo 01b]	=366		=4760				=804468			
15	406[Wil 03]										
16	480[Exo 03]										
17	576 [Exo03]	more	more	more	more						

Coherency

- There are 38 known values of $n(k, g)$ for $k \leq 14$.
- Of the known cages, seven are coherent:
  - the $(3, 5)$-cage (Petersen graph on 10 vertices),
  - the $(3, 6)$-cage (Heawood graph on 14 vertices),
  - the $(3, 8)$-cage (Tutte's 8-cage on 30 vertices),
  - the $(3, 12)$-cage (generalized hexagon on 126 vertices),
  - the $(6, 5)$-cage (Robertson graph on 40 vertices),
  - the $(7, 5)$-cage (Hoffman-Singleton graph on 50 vertices)
  - and the $(7, 6)$-cage (on 90 vertices).
- Of those, three are geometric, and two are the Petersen and Hoffman-Singleton graphs.
- The remaining two have non-Schurian coherent closure.
- We will look at them with more details.

Robertson graph $(6,5)$-cage

- The unique $(6,5)$-cage is known as the Robertson graph.
- It has 40 vertices, 3 more than the Moore bound $6^2+1=37$.
- A simple model for this cage: Remove a (visible) Petersen graph from the Hoffman-Singleton graph.

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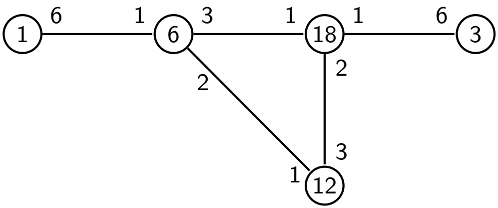
- It is a non-Schurian coherent graph.
- Its coherent closure is a rank 5 association scheme with valencies $1,6,3,12,18$.



- Its automorphism group of order 480 is of rank 7.

Baker graph $(7, 6)$-cage

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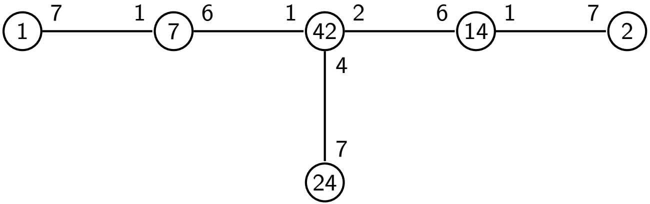
- The graph is actually the incidence graph of Baker's semiplane on 45 points.
- This cage is coherent, its coherent closure is a non Schurian scheme of rank 6 with valencies $1,7,2,14,24,42$.



- The coherent closure is non-Schurian. Its automorphism group of order 15120, the same group as the action of $3.S_7$ on 90 points as it appears in the Atlas of finite group representations.
- The group has rank 8 with valencies $1,7,1,1,7,7,24,42$.

A new construction for the Baker cage

- We start with the group $G = S_5 \oplus S_3 = \langle (0,1,2,3,4), (0,1), (5,6,7), (5,6) \rangle$.
- Let O be the orbit $O = (\{\{0,1\},\{2,3\}\},(5,6))^G$. Then $|O| = 90$.
- The association scheme $V(G,O)$ has rank 24, with valencies $1^6, 2^6, 4^6, 8^6$.
- The Baker coherent closure is a merging of this scheme.
- This gives us a different, perhaps simpler, construction for the Baker semiplane.

$n(11, 6)$

- The Moore bound for the case $k = 11$, $g = 6$ is 222.
- The existence of a Moore graph with those parameters is equivalent to the existence of a projective plane of order 10.
- Thus, the lower bound for $n(11, 6)$ is 223.
- The upper bound by the given formula is $2 \cdot 11 \cdot 11 = 242$.
- Finding a graph with more than 223 vertices and less than 242 vertices improves the upper bound, even without proof of minimality.
- Wong constructed a graph with valency 11 and girth 6 on 240 vertices.

Excision

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- Usually, the selection of the removed vertices is natural in some way.
- For example, the Robertson cage is constructed by removing a Petersen graph from the Hoffman-Singleton graph.
- Removing another Petersen, we get the $(5,5)$-cage on 30 vertices.

Infinite family of small graphs

- This infinite family includes the Wong graph on 240 vertices.
- This construction goes back to Dembowski (1968).
- We start with the Levi graph of a projective plane of order q. It has $2(q^2+q+1)$ vertices and valency $q+1$.
- We pick a pair on non-incident point and line, and remove them, as well as all $q+1$ points and $q+1$ lines incident to them.
- We are left with $2(q^2-1)$ vertices of valency q.
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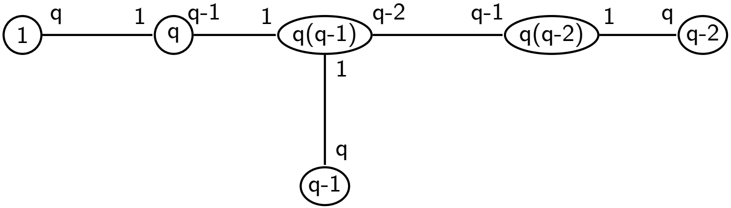
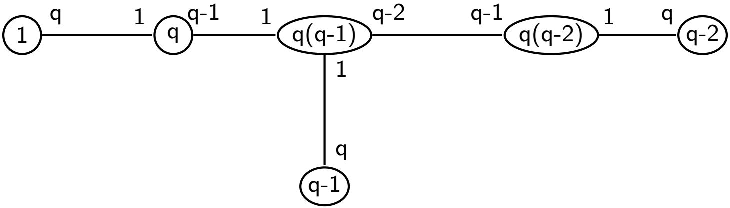
AGT view

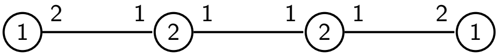
- We get an association scheme of rank 6.
- Its intersection array is:



$q = 2$



- For $q = 2$, we start from the Levi graph of Fano plane.
- The last two classes disappear.
- After the excision of 8 vertices, we are left with



- This is the hexagon.

Desarguesian planes

- Let us consider only Desarguesian projective planes.
- For a $q = p^s$, let $\Pi = PG(2, q)$ be the Desarguesian plane of order q and Δ be the Levi graph of Π, $G = Aut(\Delta)$.
- Γ is the excised graph, $H = Aut(\Gamma)$.
- $|G| = 2q^3(q^3 - 1)(q^2 - 1)s$.
- There are $(q^2 + q + 1)q^2$ anti-flags, and G is transitive on anti-flags, thus the stabilizer of an anti-flag in G has order $2q(q - 1)(q^2 - 1)s$.
- $H \leq Aut(\Gamma)$.
- H acts transitively on $V(\Gamma)$.
- H is 2-closed, thus $H = Aut(\Gamma)$.

Computational results

- For small q we calculated the group H and its rank for $q \leq 11$. The results are summarized below:

q	$\lvert H \rvert$	v	rank	Schurian
2	12	6	4	yes
3	96	16	6	yes
4	720	30	6	yes
5	960	48	10	not
7	4032	96	14	not
8	21168	126	8	not
9	23040	160	12	not
11	26400	240	22	not

Computational results

- For $q \geq 5$, we get a non-Schurian association scheme.
- Computer search reveals that for $5 \leq q \leq 11$, the centralizer algebra admits other non-Schurian mergings.
- Some of them have the same group H as automorphism group.
- Study of those extra non-Schurian mergings might prove useful in context of both AGT and EGT.

General case

- Using a computer, we calculated the structure constants of the rank 6 coherent algebra.

	1	2	3	4	5
1	q 0 0 0 0 1	0 0 0 0 0 1	0 0 0 0 q $q-2$	0 0 0 1 0 0	0 $q-1$ q $q-1$ 0 0
2	0 0 0 0 0 1	$q-1$ 0 0 0 $q-1$ 0	0 0 0 0 0 $q-2$	0 0 $q-2$ 0 0 0	0 $q-1$ 0 $q-1$ 0 0
3	0 0 0 0 q $q-2$	0 0 0 0 0 $q-2$	q^2-2q 0 0 0 q^2-3q q^2-4q+4	0 $q-2$ 0 $q-3$ 0 0	0 q^2-3q+2 q^2-2q q^2-3q+2 0 0

	1	2	3	4	5
4	0 0 0 1 0 0	0 0 $q-2$ 0 0 0	0 $q-2$ 0 $q-3$ 0 0	$q-2$ 0 0 0 $q-3$ 0	0 0 0 0 0 $q-2$
5	0 $q-1$ q $q-1$ 0 0	0 $q-1$ 0 $q-1$ 0 0	0 q^2-3q+2 q^2-2q q^2-3q+2 0 0	0 0 0 0 0 $q-2$	q^2-q 0 0 0 q^2-q q^2-2q+1

- While we only prove this for Desarguesian planes, it is also correct for the three non-Desarguesian planes of order 9.
- It seems that the proof may be extended to arbitrary projective plane.

References

- Bannai E., Ito T.: *Algebraic Combinatorics I. Association Schemes. Benjamin/Cummings, Menlo Park, 1984.*
- Exoo, Geoffrey, and Jajcay, Robert. "Dynamic cage survey." The Electronic Journal of Combinatorics DS16.
- B. Ju. Weisfeiler, A. A. Leman. A reduction of a graph to a canonical form and an algebra arising during this reduction. Nauchno - Technicheskaja Informatsia, Seria 2, **9** (1968), 12–16.

Thank you for your attention