Circulant Graphs and Jacobians

Ilya Mednykh
Sobolev Institute of Mathematics
Novosibirsk State University

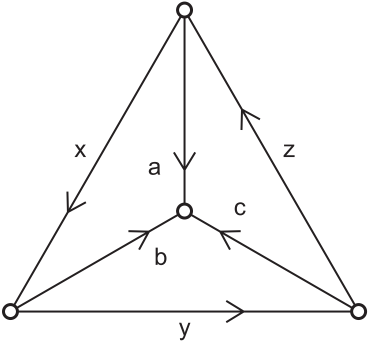
G2S2

Novosibirsk, August 15-28, 2016

This is a part of joint research with Alexander Mednykh and Madina Deryagina.

The notion of the Jacobian group of graph (also known as the Picard group, critical group, sandpile group, dollar group) was independently given by many authors (R. Cori and D. Rossin, M. Baker and S. Norine, N. L. Biggs, R. Bacher, P. de la Harpe and T. Nagnibeda). This is a very important algebraic invariant of a finite graph. In particular, the order of the Jacobian group coincides with the number of spanning trees for a graph. The latter number is known for many large families of graphs. But the structure of Jacobian for such families are still unknown. The aim of the present presentation provide structure theorems for Jacobians of circulant graphs.
The Jacobian for graphs can be considered as a natural discrete analogue of Jacobian for Riemann surfaces.
Also there is a close connection between the Jacobian and Laplacian operator for graphs.

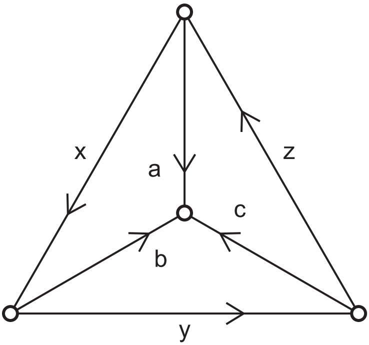
We define Jacobian $Jac(G)$ of a graph G as the Abelian group generated by flows satisfying the first and the second Kirchhoff laws. We illustrate this notion on the following simple example.



Complete graph K_4

The first Kirchhoff law is given by the equations

$$L_1 : \quad \begin{cases} a+b+c=0; \\ x-y-b=0; \\ y-z-c=0; \\ z-x-a=0. \end{cases}$$



Complete graph K_4

The second Kirchhoff law is given by the equations

$$L_2: \qquad \begin{cases} x + b - a = 0; \\ y + c - b = 0; \\ z + a - c = 0. \end{cases}$$

Now $Jac(K_4) = \langle a,\ b,\ c,\ x,\ y,\ z : L_1,\ L_2\rangle$.
Since by $L_2 : x = a - b,\ y = b - c,\ z = c - a$ we obtain

$$\langle a,\ b,\ c : a+b+c = 0, a+b+c-4b = 0, a+b+c-4c = 0, a+b+c-4a = 0\rangle =$$

$$\langle a,\ b,\ c : a + b + c = 0,\ 4a = 0,\ 4b = 0,\ 4c = 0\rangle =$$

$$\langle a,\ b : 4a = 0,\ 4b = 0\rangle \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$

So we have $Jac(K_4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

The graphs under consideration are supposed to be unoriented and finite. They may have loops, multiple edges and to be disconnected.
Let a_{uv} be the number of edges between two given vertices u and v of G.
The matrix $A = A(G) = [a_{uv}]_{u,v \in V(G)}$, is called the *adjacency matrix* of the graph G.
Let $d(v)$ denote the degree of $v \in V(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$.
The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G.
It should be noted that loops have no influence on $L(G)$. The matrix $L(G)$ is sometimes called the *Kirchhoff matrix* of G.
It should be mentioned here that the rows and columns of graph matrices are indexed by the vertices of the graph, their order being unimportant.

Recall a wide known theorem about the structure of an arbitrary Abelian group.

Let $\mathcal{A}$ be a finite Abelian group generated by $x_1, x_2, \ldots, x_n$ and satisfying the system of relations

$$\sum_{j=1}^{n} a_{ij}x_j = 0, \; i = 1, \ldots, m,$$

where $A = \{a_{ij}\}$ is an integer $m \times n$ matrix. Set $d_j, \; j = 1, \ldots, r$, for the greatest common divisor of all $j \times j$ minors of A. Then,

$$\mathcal{A} \cong \mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2/d_1} \oplus \mathbb{Z}_{d_3/d_2} \oplus \cdots \oplus \mathbb{Z}_{d_r/d_{r-1}}.$$

This is so called the Smith Normal Form of group $\mathcal{A}$.

Consider the Laplacian matrix $L(G)$ as a homomorphism $\mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$, where $|V| = |V(G)|$ is the number of vertices of G. Then $\mathrm{coker}(L(G)) = \mathbb{Z}^{|V|}/\mathrm{im(L(G))}$ is an abelian group. Let

$$\mathrm{coker}(L(G)) \cong \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|}},$$

be its Smith normal form satisfying $t_i \big| t_{i+1}$, $(1 \leq i \leq |V|)$. If graph G is connected then the groups $\mathbb{Z}_{t_1}, \mathbb{Z}_{t_1}, \ldots \mathbb{Z}_{t_{|V|-1}}$ are finite and $\mathbb{Z}_{t_{|V|}} = \mathbb{Z}$. In this case,

$$Jac(G) = \mathbb{Z}_{t_1} \oplus \mathbb{Z}_{t_2} \oplus \cdots \oplus \mathbb{Z}_{t_{|V|-1}}$$

is the Jacobian group of the graph G.
Equivalently $\mathrm{coker}(L(G)) \cong Jac(G) \oplus \mathbb{Z}$
or
$Jac(G)$ is the torsion part of cokernel of $L(G)$.

Circulant graphs

Circulant graphs can be described in several equivalent ways:

(a) The graph has an adjacency matrix that is a circulant matrix.

(b) The automorphism group of the graph includes a cyclic subgroup that acts transitively on the graph's vertices.

(c) The n vertices of the graph can be numbered from 0 to $n-1$ in such a way that, if some two vertices numbered x and y are adjacent, then every two vertices numbered z and $(z-x+y) \mod n$ are adjacent.

(d) The graph can be drawn (possibly with crossings) so that its vertices lie on the corners of a regular polygon, and every rotational symmetry of the polygon is also a symmetry of the drawing.

(e) The graph is a Cayley graph of a cyclic group.

Examples

(a) The circulant graph $C_n(s_1, \ldots, s_k)$ with jumps $s_1, \ldots, s_k$ is defined as the graph with n vertices labeled $0, 1, \ldots, n-1$ where each vertex i is adjacent to $2k$ vertices $i \pm s_1, \ldots, i \pm s_k \mod n$.

(b) n-cycle graph $C_n = C_n(1)$.

(c) n-antiprism graph $C_{2n}(1, 2)$.

(d) n-prism graph $Y_n = C_{2n}(2, n)$, n odd.

(e) The Moebius ladder graph $M_n = C_{2n}(1, n)$.

(f) The complete graph $K_n = C_n(1, 2, \cdots, [\frac{n}{2}])$.

(g) The complete bipartite graph $K_{n,n} = C_n(1, 3, \cdots, 2[\frac{n}{2}] + 1)$.

Circulant Graphs and Jacobians

The simplest possible circulant graphs with even degree of vertices are cyclic graphs $C_n = C_n(1)$. Their Jacobians are cyclic groups $\mathbb{Z}_n$. The next representative of circulant graph is a graph $C_n(1,2)$. The structure of its Jacobian is given by the following theorem.

Theorem (Structure of $Jac(C_n(1,2))$)

Let $\mathcal{A}$ be the following matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -1 & 4 & -1 \end{pmatrix}$$

Then Jacobian of the circulant graph $C_n(1,2)$ is isomorphic to the torsion part of cokernel of the operator

$$\mathcal{A}^n - I_4 : \mathbb{Z}^4 \to \mathbb{Z}^4.$$

The following corollary is straightforward consequence of the previous theorem.

Corollary

Jacobian of the graph $C_n(1,2)$ *is isomorphic to* $\mathbb{Z}_{(n,F_n)} \oplus \mathbb{Z}_{F_n} \oplus \mathbb{Z}_{[n,F_n]}$, *where* $(a,b) = GCD(a,b), [a,b] = LCM(a,b)$ *and* F_n *- Fibonacci numbers defined by recursion* $F_1 = 1,\ F_2 = 1,\ F_{n+2} = F_{n+1} + F_n,\ n \geq 1.$

Similar results can be obtained also for graphs $C_n(1,3)$ and $C_n(2,3)$. In these cases the structure of the Jacobians is expressed in terms of of real and imaginary parts of the Chebyshev polynomials $T_n(\frac{1+i}{2}),\ U_{n-1}(\frac{1+i}{2})$ and $T_n(\frac{3+i\sqrt{3}}{4}),\ U_{n-1}(\frac{3+i\sqrt{3}}{4})$ respectively.

Theorem

Jacobian $Jac(C_n(1,3))$ *is isomorphic to* $\mathbb{Z}_{d_1} \oplus \mathbb{Z}_{d_2} \oplus \cdots \oplus \mathbb{Z}_{d_5}$, *where* $d_i \big| d_{i+1}$, $(1 \leq i \leq 5)$. *Here* $d_1 = (n,\, d)$, $d_2 = d$, *if* 4 *is not divisor of* n; *otherwise* $d_1 = (n,\, d)/2$, $d_2 = d/2$, *if* $n/4$ *is even and* $d_1 = (n,\, d)/4$, $d_2 = d/4$, *if* $n/4$ *is odd. Set* $d = \mathcal{GCD}(s, t, u, v)$ *and* s, t, u, v *are integers defined by the equations* $s + i\,t = 2T_n(\frac{1+i}{2}) - 2$ *and* $u + i\,v = U_{n-1}(\frac{1+i}{2})$. *Moreover, the order of the group* $Jac(C_n(1,3))$ *is equal to* $n(s^2 + t^2)/10$.

Remark

In the above theorem the numbers d_i, $(3 \leq i \leq 5)$ can be expressed through n, s, t, u, v. But the respective formulas are rather large and complicated.

n	Jacobian $Jac(C_n(1,3))$
7	$\mathbb{Z}_{13} \oplus \mathbb{Z}_{91}$
8	$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{16}$
9	$\mathbb{Z}_{37} \oplus \mathbb{Z}_{333}$
10	$\mathbb{Z}_3 \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_{60}$
11	$\mathbb{Z}_{109} \oplus \mathbb{Z}_{1199}$
12	$\mathbb{Z}_2 \oplus \mathbb{Z}_{130} \oplus \mathbb{Z}_{1560}$
13	$\mathbb{Z}_{313} \oplus \mathbb{Z}_{4069}$
14	$\mathbb{Z}_{337} \oplus \mathbb{Z}_{10556}$
15	$\mathbb{Z}_5 \oplus \mathbb{Z}_{905} \oplus \mathbb{Z}_{2715}$
16	$\mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{136} \oplus \mathbb{Z}_{544}$
17	$\mathbb{Z}_{21617} \oplus \mathbb{Z}_{44489}$
18	$\mathbb{Z}_{3145} \oplus \mathbb{Z}_{113220}$
19	$\mathbb{Z}_{7561} \oplus \mathbb{Z}_{143659}$
20	$\mathbb{Z}_3 \oplus \mathbb{Z}_{30} \oplus \mathbb{Z}_{3030} \oplus \mathbb{Z}_{12120}$
21	$\mathbb{Z}_{41} \oplus \mathbb{Z}_{41} \oplus \mathbb{Z}_{533} \oplus \mathbb{Z}_{11193}$
22	$\mathbb{Z}_{26269} \oplus \mathbb{Z}_{1155836}$
23	$\mathbb{Z}_{63157} \oplus \mathbb{Z}_{1452611}$

n	Jacobian $Jac(C_n(1,3))$
24	$\mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{18980} \oplus \mathbb{Z}_{227760}$
25	$\mathbb{Z}_7 \oplus \mathbb{Z}_{175} \oplus \mathbb{Z}_{26075} \oplus \mathbb{Z}_{26075}$
26	$\mathbb{Z}_{219413} \oplus \mathbb{Z}_{11409476}$
27	$\mathbb{Z}_{527509} \oplus \mathbb{Z}_{14242743}$
28	$\mathbb{Z}_{29} \oplus \mathbb{Z}_{58} \oplus \mathbb{Z}_{21866} \oplus \mathbb{Z}_{612248}$
29	$\mathbb{Z}_{1524529} \oplus \mathbb{Z}_{44211341}$
30	$\mathbb{Z}_{15} \oplus \mathbb{Z}_{45} \oplus \mathbb{Z}_{45} \oplus \mathbb{Z}_{40725} \oplus \mathbb{Z}_{162900}$
31	$\mathbb{Z}_{4405969} \oplus \mathbb{Z}_{136585039}$
32	$\mathbb{Z}_{16} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{16} \oplus \mathbb{Z}_{331024} \oplus \mathbb{Z}_{1324096}$
33	$\mathbb{Z}_{12733489} \oplus \mathbb{Z}_{420205137}$
34	$\mathbb{Z}_{15306833} \oplus \mathbb{Z}_{1040864644}$
35	$\mathbb{Z}_5 \oplus \mathbb{Z}_{36800465} \oplus \mathbb{Z}_{257603255}$
36	$\mathbb{Z}_2 \oplus \mathbb{Z}_{44237570} \oplus \mathbb{Z}_{1592552520}$
37	$\mathbb{Z}_{106355317} \oplus \mathbb{Z}_{3935146729}$
38	$\mathbb{Z}_{127848949} \oplus \mathbb{Z}_{9716520124}$
39	$\mathbb{Z}_{307372573} \oplus \mathbb{Z}_{11987530347}$
40	$\mathbb{Z}_4 \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{60} \oplus \mathbb{Z}_{30790860} \oplus \mathbb{Z}_{123163440}$

More general result is given by the following theorem.

Theorem (Structure of the $Jac(C_n(1,2,\ldots,k))$)

Let $C_n(1,2,\ldots,k),\ k<\frac{n}{2}$ be the circulant graph with even degree of vertices. Let $\mathcal{A}$ be $2k\times 2k$ matrix of the form $\mathcal{A}=\left(\begin{array}{c|c} 0 & I_{2k-1} \\ \hline \multicolumn{2}{c}{\ell} \end{array}\right)$, where line ℓ has the following type $(\underbrace{-1,-1,\ldots,-1}_{k\text{ times}},2k,\underbrace{-1,-1,\ldots,-1}_{k-1\text{ times}})$, Then Jacobian of the graph $C_n(1,2,\ldots,k)$ isomorphic to the torsion part of cokernel of the linear operator

$$\mathcal{A}^n-I_{2k}:\mathbb{Z}^{2k}\to\mathbb{Z}^{2k}.$$

One of the simplest examples of circulant graph with odd degree of vertices is the Moebius ladder graph $C_{2n}(1, n)$.

Consider $4k \times 4k$ matrix $\mathcal{A}_k = \left(\begin{array}{c|c} 0 & I_{4k-1} \\ \hline \multicolumn{2}{c}{\ell_k} \end{array} \right)$, where last line ℓ_k has the following form:

$$\{-1, \ldots, -k, 3k+3, \ldots, 2k+4, -4k^2-6k, 2k+4, \ldots, 3k+3, -k, \ldots, -2\}$$

if $k \geq 2$ and $\ell_1 = \{-1, 6, -10, 6\}$ when $k = 1$.
Then the next theorem is true.

Theorem (Structure of the $Jac(C_{2n}(1, n))$)

Jacobian of the Moebius ladder graph $C_{2n}(1, n)$ is isomorphic to the torsion of cokernel of linear operator $P(\mathcal{A}) + \mathcal{A}^n : \mathbb{Z}^4 \to \mathbb{Z}^4$, where $\mathcal{A} = \mathcal{A}_1$ and $P(x) = 4 - 15x + 16x^2 - 7x^3 + x^4$.

Corollary

Jacobian $Jac(M(n))$ of the Moebius band $M(n)$ is isomorphic to $\mathbb{Z}_{(n,H_m)} \oplus \mathbb{Z}_{H_m} \oplus \mathbb{Z}_{3\{n,H_m\}}$, if $n = 2m+1$ is odd, $\mathbb{Z}_{(n,T_m)} \oplus \mathbb{Z}_{T_m} \oplus \mathbb{Z}_{2\{n,T_m\}}$, $n = 2m$ and m is even, and $\mathbb{Z}_{(n,T_m)/2} \oplus \mathbb{Z}_{2\,T_m} \oplus \mathbb{Z}_{2\{n,T_m\}}$, if $n = 2m$ and m is odd, where $(l,m) = GCD(l,m)$, $\{l,m\} = LCM(l,m)$, $H_m = T_m + U_{m-1}$, and $T_m = T_m(2)$, $U_{m-1} = U_{m-1}(2)$ are the Chebyshev polynomials of the first and the second type respectively.

This corollary can be considered as a refined version of the results obtained earlier by P. Cheng, Y. Hou, C. Woo (2006) and I.A. Mednykh, M.A. Deryagina (2011).

The following theorems are also true.

Theorem (Structure of the $Jac(C_{2n}(1,2,n))$)

Jacobian of the circulant graph $C_{2n}(1,2,n)$ is isomorphic to the torsion part of cokernel of the operator $P(\mathcal{A})+\mathcal{A}^n:\mathbb{Z}^8\to\mathbb{Z}^8$, where $\mathcal{A}=\mathcal{A}_2$ and $P(x)=6-35x^2+36x^3+x^4-11x^5+x^6+x^7$.

Theorem (Structure of the $Jac(C_{2n}(1,2,3,n))$)

Jacobian of the circulant graph $C_{2n}(1,2,3,n)$ is isomorphic to the torsion part of cokernel of the operator $P(\mathcal{A})+\mathcal{A}^n:\mathbb{Z}^{12}\to\mathbb{Z}^{12}$, where $\mathcal{A}=\mathcal{A}_3$ and $P(x)=8-63x^3+64x^4+x^5+x^6-15x^7+x^8+x^9+x^{10}$.

Circulant Graphs and Jacobians

The next theorem is a generalisation of the previous theorems. It gives the structure of Jacobian for the circulant graph $C_{2n}(1, 2, \ldots, k, n)$.

Theorem

Jacibian of the circulant graph $C_{2n}(1, 2, 3, \ldots, k, n)$ is isomorphic to the torsion part of cokernel of the operator $P(\mathcal{A}) + \mathcal{A}^n : \mathbb{Z}^{4k} \to \mathbb{Z}^{4k}$, where

$$P(x) = (2k+2) - (4(k+1)^2 - 1)x^k + 4(k+1)^2 x^{k+1} +$$

$$x^{k+2} + \ldots + x^{2k} - (4k+3)x^{2k+1} + x^{2k+2} + \ldots + x^{3k+1},$$

and $\mathcal{A} = \left(\begin{array}{c|c} 0 & I_{4k-1} \\ \hline \multicolumn{2}{c}{\ell_k} \end{array} \right)$ *- integer $4k \times 4k$–matrix with the last line*

$$\ell_k = \{-1, -2, \ldots, -k, 3k+3, 3k+2, \ldots, 2k+4, -4k^2 - 6k,$$

$$2k+4, \ldots, 3k+3, -k, -k+1, \ldots, -2\}.$$