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Strongly regular graphs

A SRG($v, k, \lambda, \mu$) is a simple undirected $k$-regular graph with $v$ vertices such that

- two adjacent vertices have $\lambda$ common neighbors,
- two non-adjacent vertices have $\mu$ common neighbors.
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Example: The Petersen graph.

$$\text{Petersen} = L(K_5) = T(5) = J(5, 2)$$

is a SRG(10,3,0,1).

(line graph of complete, triangular, Johnson)
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(line graph of complete, triangular, Johnson)
\[ L(K_6) = T(6) = J(6, 2) \quad \binom{6}{2} = 15 \]
\[ \text{SRG}(\nu = 15, \ k = 8, \ \lambda = 4, \ \mu = 4) \]
Let $A$ be the adjacency matrix, and let $C \subseteq \mathbb{F}_2^{15}$ be the span of the row vectors of $A$ over $\mathbb{F}_2$.
Then dim $C = 4$. 

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Then $\dim C = 4$.

In general, for $T(n)$,
- Tonchev (1988), Brouwer-van Eijl (1992): $\dim C$,
The weight $\text{wt}(x)$ of a vector $x \in \mathbb{F}_2^n$ is the number of 1’s in its entries: $\text{wt}(1, 1, 0, 1, 0) = 3$. 
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We say that a vector $x \in \mathbb{F}_2^n$ is

- **even** $\iff \text{wt}(x) \equiv 0 \pmod{2}$
- **doubly even** $\iff \text{wt}(x) \equiv 0 \pmod{4}$
- **triply even** $\iff \text{wt}(x) \equiv 0 \pmod{8}$
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$$\begin{align*}
ev\text{en} &\iff x \text{ is even } \quad (\forall x \in C) \\
doubly\text{ even} &\iff x \text{ is doubly even } \quad (\forall x \in C) \\
triply\text{ even} &\iff x \text{ is triply even } \quad (\forall x \in C)
\end{align*}$$

If $C$ is generated by a set of vectors $r_1, \ldots, r_k$, then
A binary linear code $C$ of length $n$ is a linear subspace of $\mathbb{F}_2^n$.

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- **even** if $\forall x \in C$ ($x$ is even)
- **doubly even** if $\forall x \in C$ ($x$ is doubly even)
- **triply even** if $\forall x \in C$ ($x$ is triply even)

If $C$ is generated by a set of vectors $r_1, \ldots, r_k$, then $C$ is even iff,

(i) $r_i$ is even for all $i \in \{1, \ldots, k\}$. 
Even, doubly even, and triply even codes

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If $C$ is generated by a set of vectors $r_1, \ldots, r_k$, then $C$ is doubly even iff,

(i) $r_i$ is doubly even for all $i \in \{1, \ldots, k\}$,
(ii) $\text{wt}(r_i \ast r_j) \equiv 0 \ (\text{mod} \ 2)$ for all $i, j \in \{1, \ldots, k\}$.

(denoting by $\ast$ the entrywise product)
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This is because the mapping

$$f : \{\text{even vectors in } \mathbb{F}_2^n\} \to \mathbb{F}_2$$

defined by

$$f : x \mapsto \frac{\text{wt}(x)}{2} \pmod{2}$$

is a quadratic form.

$$f(\sum_{i=1}^{k} a_i r_i) = \sum_{i=1}^{k} a_i^2 f(r_i) + \sum_{i<j} a_i a_j \text{wt}(r_i \ast r_j).$$
Let $A$ be the adjacency matrix, and let $C \subset \mathbb{F}_2^{15}$ be the span of the row vectors $r_1, \ldots, r_{15}$ of $A$ over $\mathbb{F}_2$. 

(i) $\text{wt}(r_i) = 8 \pmod{4}$, so $r_i$ is doubly even.

(ii) $\text{wt}(r_i \leftrightarrow r_j) = 4 \pmod{2}$ for all $i, j \in \{1, \ldots, 15\}$ with $i \neq j$.

So $C$ is doubly even.

Property too strong for the conclusion? Do these property imply $C$ is triply even? No, in general. We need:

(iii) $\text{wt}(r_h \leftrightarrow r_i \leftrightarrow r_j) = 0 \pmod{2}$ for all $h, i, j \in \{1, \ldots, k\}$.

The number of common neighbors of three vertices $\equiv 0 \pmod{2}$.
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The number of common neighbors of three vertices \( \equiv 0 \pmod{2} \)
$L(K_6) = T(6) = J(6, 2)$  \quad SRG(15, 8, 4, 4)$T(4n+2)$
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$\dim C = n - 2$. 

Note

$$rate = \frac{\dim C}{\text{length}} = \frac{n - 2}{\binom{4n+2}{2}} \rightarrow 0 \quad (n \rightarrow \infty).$$
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**Theorem (Betsumiya–M., 2012)**

The code $C$ generated by the row vectors of the adjacency matrix of $T(4n + 2)$ is a maximal triply even code.
Code of $T(4n + 2)$ is triply even

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Sharp contrast with doubly even codes: rate is always $\approx 1/2$. There are triply even codes with rate $1/4$ whenever $n \equiv 0 \pmod{16}$. 
Theorem (Betsumiya–M., 2012)

There are 10 maximal triply even codes of length 48.

9 comes from the classification of doubly even codes of length 24 classified by Pless–Sloane (1975), and the code of $T(10)$ extended by the all-one vector is the only other code.
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Motivation comes from Framed Vertex Operator Algebras (FVOA).

- The moonshine module $V^{\frac{1}{2}}$ has Virasoro frames, and each Virasoro frame gives rise to a triply even code of length 48.
- Lam–Yamauchi (2008) showed that, conversely, every triply even code of length divisible by 16 is obtained from some FVOA.
- The classification lead Lam and Shimakura to discover new FVOA≈CFT conjectured by Schellekens (1993).
$S_6 \cong Sp(4, 2) \cong PGO_4^-(3)$
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Let $V = \mathbb{F}_3^4$ be equipped with a nondegenerate quadratic form with Witt index 1, for example

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Then $|X| = 15 = \binom{6}{2}$. 

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$$\{ \pm x \} \sim \{ \pm y \}$$ 

$\iff$ $Q(x \pm y) = 0$ 

$\iff$ the line through $\langle x \rangle$ and $\langle y \rangle$ is a “tangent” 

$\iff$ the line $\langle x, y \rangle$ and the surface $Q = 0$ in $PG(3, 3)$ has exactly one point in common.
$PGO_4^- (q)$, $q$ an odd prime power

Let $V = \mathbb{F}_q^4$ be equipped with a nondegenerate quadratic form with Witt index 1, for example, with $\eta \notin (\mathbb{F}_q^\times)^2$,

$$Q(x_1, x_2, x_3, x_4) = x_1x_2 + x_3^2 - \eta x_4^2,$$

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Then $|X| = q(q^2 + 1)/2$. Adjacency by tangent. Not SRG unless $q = 3$. Brouwer–Cohen–Neumaier, Section 12.2 shows this is a $3$-class association scheme.

**Theorem (Betsumiya–M.)**

*For any odd prime power $q$, the code of this graph is triply even, of dimension at least $(q^2 - 1)/2$.***
Proof: $k, \lambda, \mu$ (BCN, Section 12.2)

Let $V = \mathbb{F}_q^4$ be equipped with a nondegenerate quadratic form $Q$ with Witt index 1. Define a graph $\Gamma$ whose vertex set is

$$X = \{\{\pm x\} \mid Q(x) = 1\},$$

with adjacency

$$\{\pm x\} \sim \{\pm y\} \iff B(x, y) = \pm 1,$$

where

$$B(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)).$$
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The graph has valency \( q^2 - 1 \equiv 0 \pmod{8} \), \( \lambda = 2(q - 1) \equiv 0 \pmod{4} \), \( \mu = 2(q - 1) \) or \( 2(q + 1) \equiv 0 \pmod{4} \), depending on \( \langle x, y \rangle \) is external or secant.
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Proof: the number of common neighbors of three vertices is even

Let $\langle x_1 \rangle, \langle x_2 \rangle, \langle x_3 \rangle$ be distinct vertices. Their common neighbors are

$$\{\langle z \rangle \mid Q(z) = 1, B(x_i, z) = \pm 1 \ (i = 1, 2, 3)\}.$$
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For simplicity, assume \( W = \langle x_1, x_2, x_3 \rangle \) is a nondegenerate 3-dimensional subspace, and consider

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Dimension $\geq (q^2 - 1)/2$

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$$\{\pm x\} \sim \{\pm y\} \iff B(x, y) = \pm 1,$$

We claim $\Gamma$ has induced $\frac{q+1}{2} K_{q-1}$. Write $V = V_+ \oplus V_-$, where $\dim V_\pm = 2$, $V_+$ contains a nonzero vector $x$ with $Q(x) = 0$, $V_-$ is anisotropic. The following subset of vertices induces $\frac{q+1}{2} K_{q-1}$:

$$Y = \{\langle \lambda x + y \rangle \mid \lambda \in \mathbb{F}_q^\times, y \in V_-, Q(y) = 1\}$$

$$= \{\langle \lambda x + y_i \rangle \mid \lambda \in \mathbb{F}_q^\times, 1 \leq i \leq (q + 1)/2\},$$

since

$$B(\lambda x + y_i, \mu x + y_j) = B(y_i, y_j) = \begin{cases} 1 & \text{if } i = j, \\ \text{not } \pm 1 & \text{otherwise.} \end{cases}$$
Maximality?

Since $\dim C \geq (q^2 - 1)/2$, the rate is at least

$$\frac{\frac{q^2 - 1}{2}}{\frac{q(q^2 + 1)}{2}} = \frac{q^2 - 1}{q(q^2 + 1)} \to 0 \quad (q \to \infty).$$

Are they maximal?
Maximality?

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\[
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\]

Are they maximal?

cf. For \( T(4n + 2) \), the rate is

\[
\frac{n - 2}{\binom{4n+2}{2}}.
\]
Maximality?

Since $\dim C \geq (q^2 - 1)/2$, the rate is at least

$$\frac{\frac{q^2 - 1}{2}}{q(q^2 + 1)/2} = \frac{q^2 - 1}{q(q^2 + 1)} \to 0 \quad (q \to \infty).$$

Are they maximal?

cf. For $T(4n + 2)$, the rate is

$$\frac{n - 2}{\binom{4n+2}{2}}.$$

Thank you for your attention!