A q and q, t-analogue of Hook Immanantal Inequalities and Hadamard Inequality for Trees

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Normalized Immanant

Let $A = (a_{i,j})_{1\leq i,j\leq n}$ be an $n \times n$ matrix. Let χ_λ be the irreducible character of the symmetric group $\mathfrak{S}_n$ indexed by the partition λ of n ($\lambda \vdash n$). Define the immanant

$$d_\lambda(A) = \sum_{\psi \in \mathfrak{S}_n} \chi_\lambda(\psi) \prod_{i=1}^{n} a_{i,\psi_i}$$

the normalized immanant

$$\overline{d}_\lambda(A) = \frac{1}{\chi_\lambda(\mathrm{id})} d_\lambda(A) = \frac{1}{\chi_\lambda(\mathrm{id})} \sum_{\psi \in \mathfrak{S}_n} \chi_\lambda(\psi) \prod_{i=1}^{n} a_{i,\psi_i}$$

where id is the identity permutation of $\mathfrak{S}_n$.

$$\overline{d}_{1^n}(A) = \sum_{\psi \in \mathfrak{S}_n} sgn(\psi) \prod_{i=1}^{n} a_{i,\psi_i} = \det A$$

$$\overline{d}_n(A) = \sum_{\psi \in \mathfrak{S}_n} 1. \prod_{i=1}^{n} a_{i,\psi_i} = \mathrm{perm}(A)$$

Lieb Conjecture

1. Schur [10] showed that for Hermitian positive semidefinite matrices A, $\overline{d}_\lambda(A) \geq \det A$, for all partitions $\lambda \vdash n$.
2. Lieb [8] conjectured for Hermitian positive semidefinite matrices A that $\text{perm}(A) \geq \overline{d}_\lambda(A)$ for all partitions $\lambda \vdash n$.
3. Consider hook partitions $k, 1^{n-k} \vdash n$ and corresponding normalized immanant is $\overline{d}_k(A)$. Heyfron [6] proved that for all Hermitian positive semidefinite $n \times n$ matrices A,

$$\det A = \overline{d}_1(A) \leq \overline{d}_2(A) \leq \cdots \leq \overline{d}_n(A) = \text{perm}(A). \tag{1}$$

Suppose T is a tree on n vertices and $L = (\ell_{i,j})_{1 \leq i,j \leq n}$ be its Laplacian matrix. Let $\overline{d}_k(L)$ be its normalized hook immanant corresponding to the hook partition $k, 1^{n-k}$.

$$\overline{d}_k(L) = \frac{1}{\chi_k(\text{id})} \sum_{\psi \in \mathfrak{S}_n} \chi_k(\psi) \prod_{i=1}^{n} \ell_{i,\psi_i}$$

Merris Conjecture

When the graph is a tree T, Merris [9] conjectured the following stronger inequality which was proved by Chan and Lam [4].

Theorem (Chan and Lam)

Let T be a tree on n vertices and let L be its Laplacian. Then, for $2 \leq k \leq n$, the normalized immanants of L satisfy the following.

$$\overline{d}_{k-1}(L) \leq \frac{k-2}{k-1}\overline{d}_k(L).$$

For $3 \leq k \leq n$, equality holds if and only if T is a star.

Problem

Let T be a tree on n vertices and $\mathcal{L}_q = ((l_q)_{ij})$ be its q**-Laplacian matrix** defined as follows:

$$(l_q)_{ij} = \begin{cases} 1 + q^2(d_i - 1) & \text{if } i = j \\ -q & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

Clearly $\mathcal{L}_q = I + q^2(D - I) - qA$. For a tree T, Bapat, Lal and Pati [1] showed that $\det(\mathcal{L}_q) = 1 - q^2$ and the matrix $\mathcal{L}_q$ is positive semidefinite when $|q| \leq 1$. When $|q| > 1$, the matrix $\mathcal{L}_q$ has exactly one negative eigenvalue.

Definition of $\mathcal{L}_{q,t}$: $\mathcal{L}_{q,t} = I + qt(D - I) - qA_1 - tA_2$

Question: Can we generalize above inequalities (Chan and Lam result) to the matrix $\mathcal{L}_q$, $\forall q \in \mathbb{R}$ and to the matrix $\mathcal{L}_{q,t}$, $\forall q, t \in \mathbb{R}$?

Generalization of Chan and Lam result

Theorem

Let T be a tree on n vertices and let $\mathcal{L}_q$ be its q-Laplacian. Then, for $2 \leq k \leq n$ and for all $q \in \mathbb{R}$,

$$\overline{d}_{k-1}(\mathcal{L}_q) + \frac{q^2-1}{k-1} \leq \frac{k-2}{k-1}\overline{d}_k(\mathcal{L}_q).$$

Moreover, for $3 \leq k \leq n$, equality holds if and only if T is a star.

Theorem

Let T be a tree on $n \geq 2$ vertices and let $\mathcal{L}_q$ be its q-laplacian. Then for all $q \in \mathbb{R}$, the normalized immanants of $\mathcal{L}_q$ satisfy

$$\det \mathcal{L}_q = \overline{d}_1(\mathcal{L}_q) \leq \overline{d}_2(\mathcal{L}_q) \leq \cdots \leq \overline{d}_k(\mathcal{L}_q) \leq \cdots \leq \overline{d}_n(\mathcal{L}_q) = \mathrm{perm}(\mathcal{L}_q)$$

Note: Lieb Conjecture is true for $\mathcal{L}_q$, *i.e.*, $\overline{d}_\lambda(\mathcal{L}_q) \leq \overline{d}_n(\mathcal{L}_q) = \mathrm{perm}(\mathcal{L}_q)$

Hadamard-Marcus Inequality

For an $n \times n$ Hermitian positive semidefinite matrix $A = (a_{i,j})_{1 \le i,j \le n}$, Hadamard-Marcus inequality states as following

$$\det A = \overline{d}_1(A) \le h(A) = \prod_{i=1}^{n} a_{i,i} \le \overline{d}_n(A) = \mathrm{perm}(A).$$

Since Heyfron result says that

$$\det A = \overline{d}_1(A) \le \overline{d}_2(A) \le \cdots \le \overline{d}_{k-1}(A) \le \overline{d}_k(A) \le \cdots \le \overline{d}_n(A) = \mathrm{perm}(A)$$

So we are interested in knowing the integer $k(A)$ (Sandwich Index) such that

$$\overline{d}_{k(A)-1}(A) \le h(A) \le \overline{d}_{k(A)}(A)$$

Heyfron [7] showed that $k(A) \ge \min(n-2, 1+\sqrt{n-1})$ for all h.p.s.d. matrix A. If the matrix A is the Laplacian L of a tree T. What can we say about $k(L)$?

Result 2

Theorem (Chan and Ng[5])

Let T be a tree on n vertices and let m be the size of the maximum matching of T. Then,

$$\left\lceil \frac{n+1}{2} \right\rceil \leq k(L) \leq \left\lceil \frac{n}{2} + \frac{m}{3} \right\rceil. \tag{3}$$

Question: **Can we get similar bound for the matrix $\mathcal{L}_q$, $\forall q \in \mathbb{R}$? Will it be depended on q?**

Answer: **Yes, Same lower bound and a slightly weaker upper bound work for $\mathcal{L}_q$, for all $q \in \mathbb{R}$.** For any $\varepsilon > 0$

$$\left\lceil \frac{n+1}{2} \right\rceil \leq k(\mathcal{L}_q) \leq \left\lceil \frac{n}{2} + \frac{m}{3} + \varepsilon \right\rceil. \tag{4}$$

Let T be a tree with n vertices and $\mathcal{M}_j$ be the set of all j-matchings in T. Let

$$m_j = \sum_{M \in \mathcal{M}_j} \prod_{v \notin M} d_v$$

$$\{\psi \in \mathfrak{S}_n : \psi \text{ has cycle type } 2^j, 1^{n-2j} \text{ with } \prod_{i=1}^{n} (l)_{i\psi_i} \neq 0\} \leftrightarrows \{M \in \mathcal{M}_j\}$$

$$\overline{d}_k(L) = \frac{1}{\chi_k(\mathrm{id})} \sum_{\psi \in \mathfrak{S}_n} \chi_k(\psi) \prod_{i=1}^{n} (\ell)_{i,\psi_i} = \frac{1}{\chi_k(0)} \sum_{j=0}^{\lfloor n/2 \rfloor} \chi_k(j) m_j.$$

Lemma (Chan and Lam[3], 1996)

Let $\chi_k(j)$ be as defined in the earlier slide. Let $0 \leq i \leq \lfloor n/2 \rfloor$. Then,

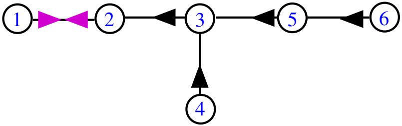
$$\sum_{j=0}^{i} \chi_k(j) \binom{i}{j} = 2^i \binom{n-i-1}{k-i-1}. \tag{5}$$

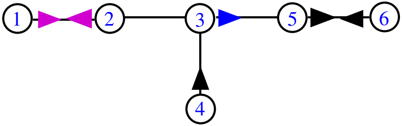
A vertex orientation is an assignment of away arrow to every vertex on its incident edge.

An edge is said to be bidirected if it has two arrows on it (one in each direction). Since T has $n-1$ edges and n vertices, every orientation will always have at least one edge which is bidirected.

Let $\mathcal{O}_i$ denote the vertex orientations of T with exactly i bidirected edges.

Let $a_i = |\mathcal{O}_i|$ be the number of vertex orientations with exactly i-bidirected edges.





Relation Between m_j and a_i

If $M \in \mathcal{M}_j$ is a j-matching, then we get several orientations $O \in \mathcal{O}_i$ for $i \geq j$. We denote this as $M \rightarrow O$.

Theorem

Let T be a tree on n vertices. Let m_j and a_i as earlier we have defined. Then, for $0 \leq j \leq \lfloor n/2 \rfloor$,

$$m_j = \sum_{i=j}^{\lfloor n/2 \rfloor} \binom{i}{j} a_i \tag{6}$$

$$a_j = \sum_{i=j}^{\lfloor n/2 \rfloor} (-1)^{i-j} \binom{i}{j} m_i \tag{7}$$

Proof.

Determine the cardinality of the set
$Q_j = \{(M, O) : M \in \mathcal{M}_j, O \in \mathcal{O}_i$ with $i \geq j$ such that $M \rightarrow O\}$ in two ways. □

Immanants in terms of a_i

Lemma

Let T be a tree with n vertices and L be its Laplacian matrix. Then

$$\overline{d}_k(L) = \sum_{i=j}^{\lfloor n/2 \rfloor} a_i 2^i \frac{\binom{n-i-1}{k-i-1}}{\binom{n-1}{k-1}} \tag{8}$$

Theorem (Chan and Lam)

Let T be a tree on n vertices and let L be its Laplacian. Then, for $2 \leq k \leq n$, the normalized immanants of L satisfy the following.

$$\overline{d}_{k-1}(L) \leq \frac{k-2}{k-1} \overline{d}_k(L).$$

For $3 \leq k \leq n$, equality holds if and only if T is a star.

Hadamard Inequality

$h(L) = m_0 = \sum_{j=0}^{\lfloor n/2 \rfloor} a_j$.

$$\begin{aligned} \overline{d}_k(L) - h(L) &= \sum_{i=0}^{m} a_i \left(2^i \frac{\binom{n-i-1}{k-i-1}}{\binom{n-1}{k-1}} - 1 \right) = \sum_{i=1}^{m} a_i [\alpha(n,k,i) - 1] \\ \text{where } \alpha(n,k,i) &= 2^i \frac{(k-1)(k-2)\cdots(k-i)}{(n-1)(n-2)\cdots(n-i)}, \text{ for } 1 \le i \le m \end{aligned}$$

Lemma

Let $\alpha(n,k,i)$ be as defined above. Then, for $1 \le i \le m$, $\alpha(n, \lfloor n/2 \rfloor, i) < 1$

Theorem

For all tree T on $n \ge 2$ vertices, $h(L) > \overline{d}_{\lfloor n/2 \rfloor}(L)$. That is $k(L) \ge \lceil (n+1)/2 \rceil$.

Upper bound

For $1 \leq i \leq m$, define $\widetilde{\alpha}_i := \alpha(n, \lceil n/2 + m/3 \rceil, i)$.

Lemma (Chan and Ng)

For fixed positive integers n and m where $1 \leq m \leq \lfloor n/2 \rfloor$, the sequence $(\widetilde{\alpha}_i)_{1 \leq i \leq m}$ is unimodal.

Corollary

The minimum value of the sequence $(\widetilde{\alpha}_i)_{1 \leq i \leq m}$ is occurs at either $\widetilde{\alpha}_1$ or $\widetilde{\alpha}_m$. Moreover for $m \geq 2$, $\widetilde{\alpha}_1 > 1$.

Proof.

For $m \geq 2$, let $\gamma = \lceil n/2 + m/3 \rceil$ then

$$\widetilde{\alpha}_1 = \frac{2(\gamma - 1)}{n-1} = \frac{2\left\lceil \frac{n}{2} + \frac{m}{3} \right\rceil - 2}{n-1} \geq \frac{n + \frac{2m}{3} - 2}{n-1} > 1$$

□

To show $\widetilde{\alpha}_m > 1$

Now we investigate when $\widetilde{\alpha}_m \geq 1$. Let $m \geq 2$ be a fixed positive integer and $n \geq 2m$ vary. Now define $\widetilde{\alpha}_{m,n} := \alpha(n, \lceil n/2 + m/3 \rceil; m)$.

Lemma

Let $m \geq 2$ be an integer. Consider the following two types of sequences:

$$A_m = \{\widetilde{\alpha}_{m,2t} : t = m, m+1, \ldots\}$$

$$B_m = \{\widetilde{\alpha}_{m,2t+1} : t = m, m+1, \ldots\}$$

Then

(a) *The sequence A_3 is increasing and all the other sequences are unimodal.*

(b) *For $r \geq 1$, the sequences B_{3r+1} are non-increasing.*

(c) *For $m \neq 3r+1$ and $r \geq 1$, the sequences B_m are strictly decreasing.*

(d) *For $m \neq 3r$ and $r \geq 1$, the sequences A_m are strictly decreasing.*

Lemma

For $n \geq 2m$, we have $\widetilde{\alpha}_{m,n} > 1$ except for $\widetilde{\alpha}_{3,2t}$ with $t \geq 3$ and $\widetilde{\alpha}_{6,12}$.

Lemma

With $\widetilde{\alpha}_i$ as defined above, for $1 \leq i \leq m$, $\widetilde{\alpha}_i > 1$, except for

1. *$\widetilde{\alpha}_3$ when $n = 2t, m = 3$ for $t \geq 3$ and*
2. *for $\widetilde{\alpha}_6$ when $n = 12, m = 6$.*

All trees satisfy $k(L) < \lceil n/2 + m/3 \rceil$ except following exceptions

1. trees with even number of vertices and with $m = 3$ and
2. trees on 12 vertices with $m = 6$.

Definition of $m_j(q)$ and $a_j(q)$

Let $\mathcal{M}_j$ be a set of all j-matching and for $0 \le j \le \lfloor n/2 \rfloor$ define

$$m_j(q) = q^{2j} \sum_{M \in \mathcal{M}_j} \prod_{v \notin M} [1 + q^2(d_v - 1)] \tag{9}$$

- Let $O \in \mathcal{O}_j$ be an j-bidirected vertex orientation and $\text{bidir}(O)$ be the set of bidirected edges in O.
- Let $\text{free}(O) = \{v \in V : v \notin \text{bidir}(O)\}$.
- Let $\text{Lexaway}(O) = 2j + 2$ [the number of vertices in $\text{free}(O)$ having away arrow from a lexicographic minimum edge in $\text{bidir}(O)$]
- Let $1 \le j \le \lfloor n/2 \rfloor$ and $M \in \mathcal{M}_j$. Define $\text{Orient}_M = \{O \in \mathcal{O}_j : M \to O\}$

Relation Between $m_j(q)$ and $a_i(q)$

$a_0(q) = 1 - q^2$ and for $j \geq 1$

$$a_j(q) = \sum_{O \in \mathcal{O}_j} q^{\text{Lexaway}(O)} = \sum_{M \in \mathcal{M}_j} \sum_{O \in \text{Orient}_M} q^{\text{Lexaway}(O)}. \qquad (10)$$

Theorem

Let T be a tree on n vertices. Let $m_j(q)$ and $a_j(q)$ be defined as earlier. Then for $0 \leq j \leq \lfloor n/2 \rfloor$,

$$m_j(q) = \sum_{i=j}^{\lfloor n/2 \rfloor} \binom{i}{j} a_i(q) \qquad (11)$$

$$a_j(q) = \sum_{i=j}^{\lfloor n/2 \rfloor} (-1)^{i-j} \binom{i}{j} m_i(q) \qquad (12)$$

Lieb conjecture for $\mathcal{L}_q$

Let T be a tree on n vertices and $\mathcal{L}_q$ be its q-Laplacian. Then

$$\overline{d}_k(\mathcal{L}_q) = \frac{1}{\chi_k(\mathrm{id})} \sum_{\psi \in \mathfrak{S}_n} \chi_k(\psi) \prod_{i=1}^{n} (\ell_q)_{i,\psi_i} = \frac{1}{\chi_k(\mathrm{id})} \sum_{j=0}^{\lfloor n/2 \rfloor} \chi_k(j) m_j(q). \tag{13}$$

Recall that $|\chi_\lambda(\sigma)| \leq |\chi_\lambda(0)|$, for all permutation $\sigma \in \mathfrak{S}_n$.

$$\overline{d}_\lambda(\mathcal{L}_q) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{\chi_\lambda(\sigma)}{\chi_\lambda(0)} m_j(q) \leq \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{|\chi_\lambda(\sigma)|}{|\chi_\lambda(0)|} m_j(q) \leq \sum_{j=0}^{\lfloor n/2 \rfloor} m_j(q) = \mathrm{perm}(\mathcal{L}_q)$$

Lemma

Let T be a tree on n vertices and $\mathcal{L}_q$ be its q-Laplacian. Then the normalized immanant $\overline{d}_k(\mathcal{L}_q)$ corresponding to the hook partition $k, 1^{n-k}$ is

$$\overline{d}_k(\mathcal{L}_q) = \sum_{i=0}^{\lfloor n/2 \rfloor} a_i(q) 2^i \frac{\binom{n-i-1}{k-i-1}}{\binom{n-1}{k-1}}.$$

Main Theorems

Theorem

Let T be a tree on n vertices and let $\mathcal{L}_q$ be its q-Laplacian. Then, for $2 \leq k \leq n$ and for all $q \in \mathbb{R}$,

$$\overline{d}_{k-1}(\mathcal{L}_q) + \frac{q^2 - 1}{k-1} \leq \frac{k-2}{k-1}\overline{d}_k(\mathcal{L}_q).$$

Moreover, for $3 \leq k \leq n$, equality holds if and only if T is a star.

Theorem

Let T be a tree on n vertices and let $\mathcal{L}_{q,t}$ be its q,t-laplacian. Then, for $2 \leq k \leq n$ and when $qt \geq 0$, the normalized immanants of $\mathcal{L}_{q,t}$ satisfy

$$\overline{d}_{k-1}(\mathcal{L}_{q,t}) + \frac{qt - 1}{k-1} \leq \frac{k-2}{k-1}\overline{d}_k(\mathcal{L}_{q,t}).$$

We have seen that $h(\mathcal{L}_q) = m_0(q) = \sum_{j=0}^{\lfloor n/2 \rfloor} a_j(q)$.

$$\begin{aligned} \overline{d}_k(\mathcal{L}_q) - h(\mathcal{L}_q) &= \sum_{i=0}^{m} a_i(q) \left(2^i \frac{\binom{n-i-1}{k-i-1}}{\binom{n-1}{k-1}} - 1 \right) = \sum_{i=1}^{m} a_i(q)[\alpha(n,k,i) - 1] \\ \text{where } \alpha(n,k,i) &= 2^i \frac{(k-1)(k-2)\cdots(k-i)}{(n-1)(n-2)\cdots(n-i)}, \text{ for } 1 \le i \le m \end{aligned}$$

For star trees with n vertices, we have $\overline{d}_k(\mathcal{L}_q) - h(\mathcal{L}_q) = q^2(2k-n-1)$.
So we consider trees other than stars

Lemma (Chan and Ng, Theorem 2.2)

Let $\alpha(n,k,i)$ *be as defined above. Then, for* $1 \le i \le m$, $\alpha(n, \lfloor n/2 \rfloor, i) < 1$

Why ε is needed?

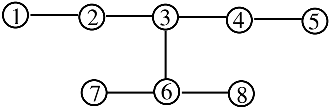
In these exceptional cases, we show that the next larger integer suffices for $\mathcal{L}_q$.
Let $k_1 = \left\lceil \frac{n}{2} + \frac{m}{3} \right\rceil$ and for $\varepsilon > 0$, let $k_2 = \left\lceil \frac{n}{2} + \frac{m}{3} + \varepsilon \right\rceil$.
For the following graph (which is an exception graph as defined in above Lemma), $k_1 = 5$ and

$$a_0 = 1 - q^2,\ a_1 = 7q^2,\ a_2 = 13q^4 + 5q^6,\ a_3 = 7q^6 + 4q^8$$

and $h(\mathcal{L}_q) = m_0(q) = 1 + 6q^2 + 13q^4 + 12q^6 + 4q^8$.

$$\overline{d}_5(\mathcal{L}_q) - h(\mathcal{L}_q) = \frac{-12q^8 + 4q^6 + 65q^4 + 35q^2}{35}.$$

Thus, for $q \geq 2$, $\overline{d}_5(\mathcal{L}_q) - h(\mathcal{L}_q) < 0$ and so it is not true that $\overline{d}_5(\mathcal{L}_q) \geq h(\mathcal{L}_q)$.



Lemma

Let T be tree on n vertices with maximum matching size m. Then $\alpha(n, k_2, m) > 1$.

Theorem

Let $\mathcal{L}_q$ be the q-laplacian matrix of a tree T on n vertices and let m be the size of the largest matching of T. Then, for all $q \in \mathbb{R}$ and for all $\varepsilon > 0$

$$\left\lceil \frac{n}{2} + \frac{1}{2} \right\rceil \leq k(\mathcal{L}_q) \leq \left\lceil \frac{n}{2} + \frac{m}{3} + \varepsilon \right\rceil. \quad (14)$$

Theorem

Let $\mathcal{L}_{q,t}$ be the q,t-laplacian matrix of a tree T on n vertices and let m be the size of the largest matching of T. Then, for all $q, t \in \mathbb{R}$ with $qt \geq 0$ and for all $q, t \in \mathbb{C}$ with $q = z$, $t = \overline{z}$, and for all $\varepsilon > 0$, the sandwich index $k(\mathcal{L}_{q,t})$ satisfies

$$\left\lceil \frac{n}{2} + \frac{1}{2} \right\rceil \leq k(\mathcal{L}_{q,t}) \leq \left\lceil \frac{n}{2} + \frac{m}{3} + \varepsilon \right\rceil. \quad (15)$$

Open Problems

1. Lieb Conjecture is proved for $n \leq 9$.
2. Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a complex matrix. If A is h.p.s.d. matrix then

$$\text{(Fischer's inequality) } \det A \leq \det(diag\{A_{11}, A_{22}\})$$

$$\text{(Grone and Merris) } d_2(A) \leq d_2(diag\{A_{11}, A_{22}\})$$

and

$$\text{(Lieb's inequality) } \text{perm}A \geq \text{perm}(diag\{A_{11}, A_{22}\})$$

THANK YOU
for your
ATTENTION!

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