A generalization of a theorem of Hoffman

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- Let $\lambda_0, \lambda_1, \ldots, \lambda_t$ be the distinct eigenvalues of G and m_i be the multiplicity of λ_i $(i = 0, 1, \ldots, t)$. Then the multiset

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- Note that $A(L(G)) + 2I = B(G)^T B(G)$, where $L(G)$ is the line graph of G and $B(G)$ is the vertex-edge-incidence matrix of G. This shows that every line graph is a generalized line graph.

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Now, we will introduce two theorems about generalized line graphs.

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Theorem (Cameron, Goethals, Seidel and Shult, 1976)

Let G be a connected graph with smallest eigenvalue at least -2. Then either G is a generalized line graph, or G has at most 36 vertices.

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The proof heavily relies on the classification of the irreducible root lattices.

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Now we give a result of Hoffman.

Theorem (Hoffman, 1977)

Let $-1-\sqrt{2} < \lambda \leq -2$ be a real number. Then there exists an integer $f(\lambda)$ such that if G is a graph with smallest eigenvalue at least λ and minimun valency at least $f(\lambda)$, then G is a generalized line graph.

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- The proof does not rely on the classification of irreducible root lattices. But you have to pay a price for it. Namely you need to assume that the minimum valency is large.
- In this talk, we will give some generalizations the theorem of Hoffman.

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Main theorem

Let $t \geq 2$ be a positive integer. Then there exists a positive integer $\kappa(t)$ such that if a graph G satisfies the following conditions:

1. $k(x) > \kappa(t)$ for all $x \in V(G)$;
2. $\bar{a}(x) \leq \frac{k(x)-\kappa(t)}{t}$ for all $x \in V(G)$;
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then the adjacency matrix A of G satisfies

$$A + (t+1)I = N^T N$$

where N is a $(0,1)$-matrix.

A geometric interpretation

- Let G be a graph with smallest eigenvalue at least $-t-1$. The meaning of this result is that if G satisfies some local condition, then G is the point graph of a partial linear space $(V(G), \mathcal{L})$ where each vertex lies in exactly $t+1$ lines.

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- And the number of triangles in a graph can be calculated using the spectrum.
- Now, we will give some examples.
- Later in the talk, I will give a more general result.

Application 1

There exists a positive integer q' such that any graph, that is cospectral with the Hamming graph $H(3, q)$, and $q \geq q'$, its adjacency matrix A satisfies

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where N is a $(0, 1)$-matrix.

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Application 2

There exists a positive integer v' such that any graph, that is cospectral with the Johnson graph $J(v,3)$, and $v \geq v'$ its adjacency matrix A satisfies

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Remarks

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- For Application 1, it can be shown that it is locally the disjoint union of $3K_{q-1}$'s. This was already shown by Bang et al.
- Moreover, they showed that for $q \geq 36$ the Hamming graph $H(3, q)$ is determined by its spectrum.
- Van Dam et al. gave two constructions to construct cospectral graphs with $J(v, 3)$. Application 2 tells us that they must come from partial linear spaces.

Hoffman graphs

We will introduce Hoffman graphs. They are very important for our proof.

Hoffman graphs, 2

Definitions

- A Hoffman graph $\mathfrak{h}$ is a pair (H, μ) of a graph $H = (V, E)$ and a labeling map $\mu : V \to \{f, s\}$, satisfying the following conditions:
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- A vertex with label s called a slim vertex;
  A vertex with label f called a fat vertex;
  $V_s = V_s(\mathfrak{h})$ the set of slim vertices of $\mathfrak{h}$;
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- If every slim vertex has a fat neighbor, we call $\mathfrak{h}$ fat;
  If every slim vertex has at least t fat neighbors, we call $\mathfrak{h}$ t-fat.

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- If every slim vertex has a fat neighbor, we call $\mathfrak{h}$ fat;
  If every slim vertex has at least t fat neighbors, we call $\mathfrak{h}$ t-fat.
- The slim graph of a Hoffman graph $\mathfrak{h}$ is the subgraph of H induced on $V_s(\mathfrak{h})$.

Special matrix

Definitions

- For a Hoffman graph $\mathfrak{h}$, let A be the adjacency matrix of H

$$A = \begin{pmatrix} A_s & C \\ C^T & O \end{pmatrix}$$

in a labeling in which the fat vertices come last.

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Note that each row and column of a special matrix is indexed by slim vertices. For $x, y \in V_s(\mathfrak{h})$, $(CC^T)_{xy}$ is the number of common fat neighbors of x and y.

Smallest eigenvalue

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Lemma

- If $\mathfrak{h}'$ is an induced Hoffman subgraph of a Hoffman graph $\mathfrak{h}$, then $\lambda_{\min}(\mathfrak{h}') \geq \lambda_{\min}(\mathfrak{h})$ holds.

Ostrowski-Hoffman limit theorem

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Ostrowski-Hoffman Theorem

Let $\mathfrak{h}$ be a Hoffman graph. Let $G(\mathfrak{h}, n)$ be the ordinary graph obtained from $\mathfrak{h}$ by replacing each fat vertex f by a slim n-clique $K_n(f)$, and joining all the neighbors of f with all the vertices of $K_n(f)$. Then

$$\lambda_{\min}(G(\mathfrak{h}, n)) \geq \lambda_{\min}(\mathfrak{h}).$$

and

$$\lim_{n\to\infty} \lambda_{\min}(G(\mathfrak{h}, n)) = \lambda_{\min}(\mathfrak{h}).$$

Structure theorem of Hoffman graphs

In this section we will give some structure theorem of Haffman graphs.

Direct Sum

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Definition

Let $\mathfrak{h}$ be a Hoffman graph and $\mathfrak{h}^1$ and $\mathfrak{h}^2$ be two induced Hoffman subgraphs of $\mathfrak{h}$. The Hoffman graph $\mathfrak{h}$ is called the direct sum of $\mathfrak{h}^1$ and $\mathfrak{h}^2$, denoted by $\mathfrak{h} = \mathfrak{h}^1 \bigoplus \mathfrak{h}^2$, if and only if $\mathfrak{h}^1, \mathfrak{h}^2$ and $\mathfrak{h}$ satisfy the following conditions:

(i) $V(\mathfrak{h}) = V(\mathfrak{h}^1) \bigcup V(\mathfrak{h}^2)$;

(ii) $\{V_s(\mathfrak{h}^1), V_s(\mathfrak{h}^2)\}$ is a partition of $V_s(\mathfrak{h})$;

(iii) if $x \in V_s(\mathfrak{h}^i)$, $f \in V_f(\mathfrak{h})$ and $x \sim f$, then $f \in V_f(\mathfrak{h}^i)$;

(iv) if $x \in V_s(\mathfrak{h}^1)$ and $y \in V_s(\mathfrak{h}^2)$, then x and y have at most one common fat neighbor, and they have exactly one common fat neighbor if and only if they are adjacent.

The main reason for this definition is that the special matrix of $\mathfrak{h}$,$S(\mathfrak{h})$, is a block matrix with blocks $S(\mathfrak{h}^1)$ and $S(\mathfrak{h}^2)$. That is,

$$S(\mathfrak{h}) = \begin{pmatrix} S(\mathfrak{h}^1) & 0 \\ 0 & S(\mathfrak{h}^2) \end{pmatrix}$$

Blackboard Example

Definition

If $\mathfrak{h} = \mathfrak{h}_1 \bigoplus \mathfrak{h}_2$ for some induced Hoffman subgraphs $\mathfrak{h}_1$ and $\mathfrak{h}_2$, then we call $\mathfrak{h}$ decomposable. Otherwise $\mathfrak{h}$ is called indecomposable.

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Definition

Let $\mathfrak{G}$ be a family of Hoffman graphs. A Hoffman graph $\mathfrak{g}$ is called a $\mathfrak{G}$-line Hoffman graph if it is an induced Hoffman subgraph of $\mathfrak{h} = \bigoplus_{i=1}^{t} \mathfrak{h}_i$ where $\mathfrak{h}_i$ is isomorphic to an induced Hoffman subgraph of some Hoffman graph in $\mathfrak{G}$ for $i = 1, \ldots, t$ such that $\mathfrak{g}$ and $\mathfrak{h}$ have the same slim graph.

A family of Hoffman graphs

Now we use the above defintions to define a family of Hoffman graphs.

Definition

Let t be a positive integer. We define $\mathfrak{G}(t)$ to be the family of pairwise non-isomorphic indecomposable t-fat Hoffman graphs with special matrix either $(-t-1)$ or $\begin{pmatrix} J_{s_1}-(t+1)I_{s_1} & -J \\ -J & J_{s_2}-(t+1)I_{s_2} \end{pmatrix}$ where $1 \leq s_1, s_2 \leq t$.

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Note that every Hoffman graph in $\mathfrak{G}(t)$ has smallest eigenvalue $-t-1$.

An important result

Let $\mathfrak{h}^{(t)}$ be the Hoffman graph with unique slim vertex adjacent to t fat vertices.

Theorem

Let t be a positive integer. Every t-fat Hoffman graph with smallest eigenvalue at least $-t-1$ is a $\mathfrak{G}(t)$-line Hoffman graph.

Some more definitions

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- The local valency at x is the quantity $\frac{|2E(\Delta(x))|}{k(x)}$ where $k(x)$ is the valency of x, and is denoted by $\bar{a}(x)$.

Main result 1

Main theorem (Local valency version)

Let $t \geq 2$ be a positive integer and $s \in \{t-1, t\}$. Then there exists a positive integer $\kappa(t)$ such that if a graph G satisfies the following conditions:

1. $k(x) > \kappa(t)$ for all $x \in V(G)$;
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(a) If $s = t-1$, then G is the slim graph of a t-fat $\mathfrak{G}(t)$-line Hoffman graph;

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We already have seen (b) before. (In quite different form.)

Main result 2

Main theorem (Plex version)

Let $t \geq 2$ be a positive integer and $s \in \{t-1, t\}$. Then there exists a positive integer $K(t)$ such that if a graph G satisfies the following conditions:

1. $k(x) > K(t)$ for all $x \in V(G)$;
2. for all $x \in V(G)$, a (t^2+1)-plex containing x has order at most $\frac{k(x)-K(t)}{s}$;
3. $\lambda_{\min}(G) \geq -t-1$,

then the following holds:

(a) If $s = t-1$, then G is the slim graph of a t-fat $\mathfrak{G}(t)$-line Hoffman graph;

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Key idea of the proof. Let G be a graph satisfies three conditions in main theorem. Then we will construct a Hoffman graph $\mathfrak{h}(G, m, n)$(Associated Hoffman graph of G) obtained from G by putting some fat vertices which correspond to very dense subgraphs of G(quasi-clique).

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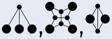
Remark. We assume $t \geq 2$, because of the second condition. For $t = 1$, we do not need the second condition. In this case, we obtain Hoffman original theorem.

Using the plex version of our main theorem and a bound a la Hoffman on the order of t-plexes, we can show:

2-clique extension of a grid

There exists a positive integer t' such that any graph, that is cospectral with the 2-clique extension of $(t_1 \times t_2)$-grid is the slim graph of a 2-fat $\{$  $\}$-line Hoffman graph for all $t_1 \geq t_2 \geq t'$.

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- For the square grid, we could also use the local valency version of our main theorem, but not for the non-square grids, as they have five distinct eigenvalues.

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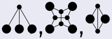
There exists a positive integer t' such that any graph, that is cospectral with the 2-clique extension of $(t_1 \times t_2)$-grid is the slim graph of a 2-fat {}-line Hoffman graph for all $t_1 \geq t_2 \geq t'$.

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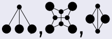
There exists a positive integer t' such that any graph, that is cospectral with the 2-clique extension of $(t_1 \times t_2)$-grid is the slim graph of a 2-fat {  }-line Hoffman graph for all $t_1 \geq t_2 \geq t'$.

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- Using this result Yang, Abiad and myself showed that the 2-clique extension of the $t \times t$-grid is determined by its spectrum if t is very large.
- This result will be used in the next talk by Sasha Gavrilyuk to show that certain Grassmann graphs are unique as distance-regular graphs.

Thank you for your attention!