On characterizations of association schemes by intersection numbers

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Coherent configurations

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- $1_\Omega = \{(\alpha, \alpha) : \alpha \in \Omega\}$ is a union of relations from S,
- S contains $s^* = \{(\alpha, \beta) : (\beta, \alpha) \in s\}$ for all $s \in S$,
- for all $r, s, t \in S$, the intersection number

$$c_{rs}^t = |\{\gamma \in \Omega : (\alpha, \gamma) \in r, \ (\gamma, \beta) \in s\}|$$

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Problem

Which coherent configurations are separable, i.e., uniquely determined by the intersection numbers?

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- the valencies n_r of basis graphs $r \in S$,
- the homogeneity, commutativity, primitivity, etc.

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Remarks

- a combinatorial isomorphism f induces an algebraic isomorphism φ by $s^\varphi := s^f$ for all $s \in S$;
- "$\mathcal{X}$ is uniquely determined by the intersection numbers" (i.e., is separable) means that any algebraic isomorphism is induced by combinatorial.

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Thus, every thin scheme is separable.

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Theorem (Muzychuk-P, 2012)

Let $\mathcal{X}$ be a quasi-thin scheme such that $m \notin \{4, 7\}$. Then

- $\mathcal{X} = \mathrm{Inv}(G)$ for some $G \leq \mathrm{Sym}(\Omega)$,
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Remarks:

- the condition $m \notin \{4, 7\}$ is essential,
- the nonhomogeneous case is open.

Pseudocyclic schemes

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Theorem (Muzychuk-P, 2012)

Let $\mathcal{X}$ be a pseudocyclic scheme such that $n > Ck^5$. Then

- $\mathcal{X} = \mathrm{Inv}(G)$, where G is a Frobenius group of order nk,
- $\mathcal{X}$ is separable.

Schemes associated with TI-subgroups

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- G is a p-group (here $\mathcal{X}$ is quasi-thin if $p = 2$).

Pseudo TI-schemes

Proposition

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- $|\Delta| \in \{1, k\}$ for all H-orbits Δ, where $k = |H|$,

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Theorem (Chen-P, 2016)

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