Minimum supports of eigenfunctions of Hamming graphs

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$\Sigma_q = \{0, 1, \ldots, q-1\}$.

Definition

The Hamming distance $d(x, y)$ between vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ from Σ_q^n is the number of positions i such that $x_i \neq y_i$.

Definition

The Hamming graph $H(n, q)$ is a graph whose vertex set is Σ_q^n and two vertices are adjacent if the Hamming distance between them equals 1.

$G = (V, E)$ — is a simple graph.

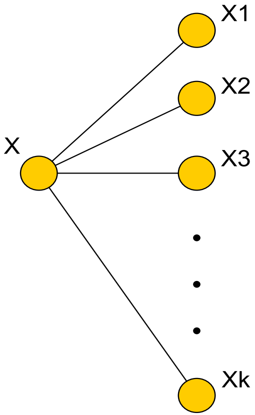
Definition

A function $f : V \longrightarrow \mathbb{R}$ is called an *eigenfunction* of the graph G corresponding to an eigenvalue λ if

$$\lambda \cdot f(x) = \sum_{y \in N(x)} f(y)$$

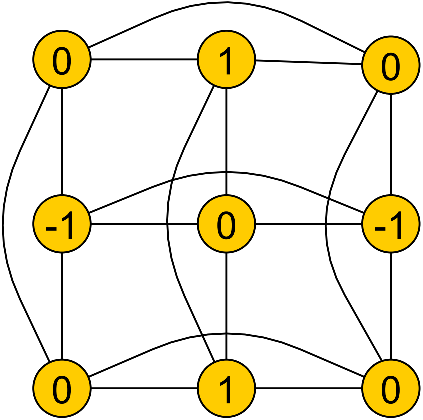
for any vertex x.

Basic definitions



$$\lambda \cdot f(x) = f(x_1) + f(x_2) + \ldots + f(x_k)$$

Example



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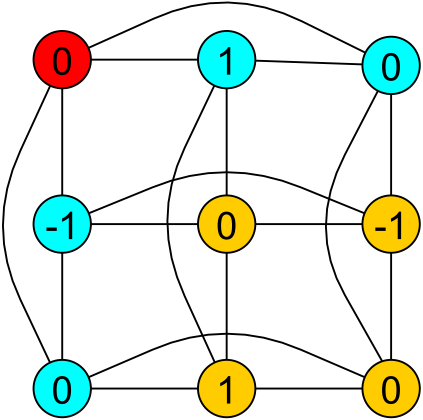

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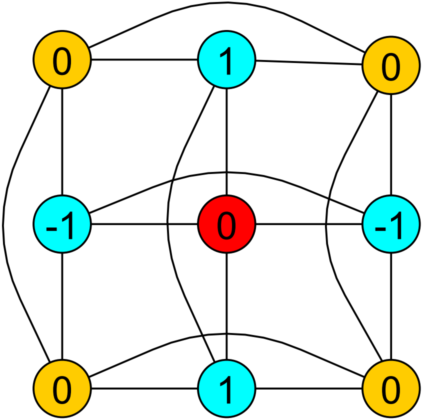

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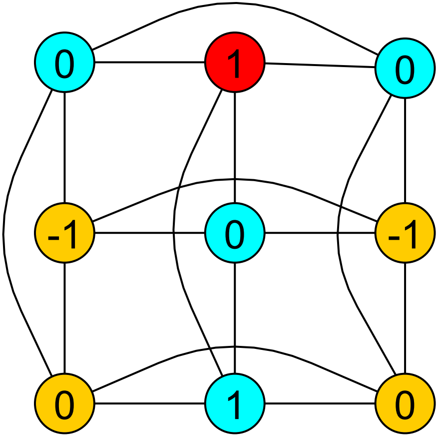

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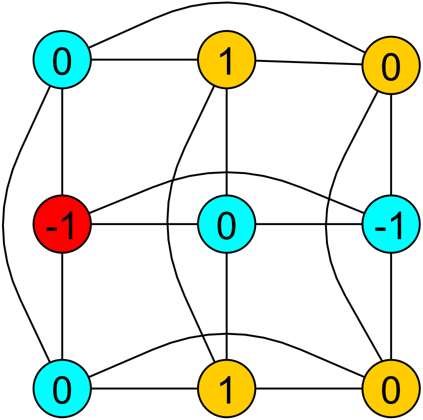

Basic definitions

$V = \{v_1, v_2, \ldots, v_n\}$.

Equivalent definition

f is an eigenfunction of G corresponding to λ if the vector $(f(v_1), \ldots, f(v_n))^T$ is an eigenvector of the adjacency matrix of G.

- A set C of vertices of a regular graph G is called a 1-perfect code iff every ball of radius 1 contains exactly one element of C.

Eigenfunctions in combinatorial configurations

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- 1–perfect code C can be considered as an eigenfunction with two values corresponding to the eigenvalue -1.
  Let $f(x) = a$ for $x \in C$ and $f(x) = b$ for $x \in G \setminus C$.
  Then $(-1)a = kb$ and $(-1)b = (k-1)b + a$.

Bitrades

Definition

A 1-perfect bitrade is a pair (T_0, T_1) of disjoint nonempty sets of vertices of $H(n,q)$ such that for every radius-1 ball B it holds $|B \cap T_0| = |B \cap T_1| = \{0,1\}$.

- Example. If C_1 and C_2 are 1–perfect codes, then the pair $(C_1 \setminus C_2, C_2 \setminus C_1)$ is a 1–perfect bitrade.
- $\chi_{(T_0,T_1)} = \chi(T_0) - \chi(T_1)$ — characteristic function of a bitrade (T_0, T_1).
  $\chi_{(T_0,T_1)} : V \longrightarrow \{-1,0,1\}$ is an eigenfunction corresponding to the eigenvalue -1.

Minimum support

Definition

The set $S(f) = \{x \in V \mid f(x) \neq 0)\}$ is called the support of f.

Problem

To find the minimum cardinality of the support of eigenfunctions and 1–perfect bitrades in the Hamming graphs.

Some results

The spectrum of $H(n,q)$ is
$\{\lambda_m = n(q-1) - qm \mid m = 0, 1, \ldots, n\}$.

Theorem (Potapov, 2012)

Let $f : H(n,q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue λ_m and $f \not\equiv 0$. Then the following statements are true:

- $|S(f)| \geq 2^m$.
- If $q = 2$, then the minimum cardinality of the support f equals $\max(2^m, 2^{n-m})$.

Some results

Theorem (Vorob'ev, Krotov, 2014)

Let $f : H(n,q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to the eigenvalue λ_m and $f \not\equiv 0$. Then

$$|S(f)| \geq 2^m (q-2)^{n-m}$$

for $\frac{mq^2}{2n(q-1)} > 2$ and

$$|S(f)| \geq q^n \left(\frac{1}{q-1}\right)^{m/2} \left(\frac{m}{n-m}\right)^{m/2} \left(1 - \frac{m}{n}\right)^{n/2}$$

for $\frac{mq^2}{2n(q-1)} \leq 2$.

Some results

Theorem (Vorob'ev, Krotov, 2014)

Let f be a 1–perfect bitrade in $H(n,q)$, $q \geq 3$ and $f \not\equiv 0$. Then

$$|S(f)| \geq 2^{n-\frac{n-1}{q}}(q-2)^{\frac{n-1}{q}}$$

for $q \geq 4$ and

$$|S(f)| \geq 3^{\frac{n}{2}}(1-\frac{1}{n})^{\frac{n}{2}}(1+\frac{3}{2(n-1)})^{\frac{2n+1}{6}}$$

for $q = 3$.

Minimum support of eigenfunction of $H(n,q)$ corresponding to λ_1

Remark

If $f : H(n,q) \longrightarrow \mathbb{R}$ is an eigenfunction corresponding to $\lambda_0 = n(q-1)$, then f is a constant.

The set of vertices $x = (x_1, x_2, \ldots, x_n)$ of $H(n,q)$ such that $x_i = k$ is denoted by $T_k(i,n)$.

Theorem (V., 2016)

Let $f : H(n,q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to λ_1, $f \not\equiv 0$ and $q > 2$. Then $|S(f)| \geq 2(q-1)q^{n-2}$. Moreover, if $|S(f)| = 2(q-1)q^{n-2}$, then

$$f(x) = \begin{cases} c, & \text{for } x \in T_k(i,n) \setminus T_m(j,n); \\ -c, & \text{for } x \in T_m(j,n) \setminus T_k(i,n); \\ 0, & \text{otherwise.} \end{cases},$$

where $c \neq 0$ is a constant, i, j, k, m are some numbers and $i \neq j$.

Main theorem

Theorem (V., 2016)

Let $f : H(3,q) \longrightarrow \mathbb{R}$ be an eigenfunction corresponding to λ_2, $f \not\equiv 0$ and $q > 4$. Then $|S(f)| \geq 4(q-1)$.

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