On Fourier decomposition of Preparata-like codes in the graph of the hypercube

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Outline

1. Notations
2. Functions on the graph of the hypercube
3. Preparata-like and perfect codes, perfect colorings
4. Fourier transform of Preparata codes

n-dimensional binary Hamming space $\mathbf{Q}^n = \{0,1\}^n$ with component-wise modulo-2 addition and the Hamming metric.
$wt(\alpha) = \sum_{i=1}^{n} |\alpha_i|$ – the Hamming weight of $\alpha \in \mathbf{Q}^n$
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vertices: $\mathbf{Q}^n$,

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Eigenvalues of the hypercube (= eigenvalues of D)

$n,\ n-2, \ldots, n-2i, \ldots, -n+2,\ -n.$

$V = \{f : \mathbf{Q}^n \longrightarrow \mathbb{C}\}$

$f \leftrightarrow (f(0,\ldots,0), f(0,\ldots,0,1), \ldots, f(1,\ldots,1))^T$

Let V_i be the eigensubspace with the eigenvalue $n-2i, \quad i = 0,1,\ldots,n$.

$V = V_0 \times V_1 \times \ldots \times V_n$

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Eigensubspace V_i (the eigenvalue $n-2i$)

Let $f^{\mathbf{a}} : \mathbf{Q}^n \longrightarrow \mathbb{C}$ such that

$f^{\mathbf{a}}(\mathbf{x}) = (-1)^{a_1x_1+\dots+a_nx_n}$

An orthogonal basis of V_i: $\quad B_i = \{f^{\mathbf{a}} \ : \ wt\mathbf{a} = i\}$

$V_0 = \{const\}$

$V_n = \{const \cdot (-1)^{x_1+\dots+x_n}\}$

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An orthogonal basis of V

$B_0 \bigcup B_1 \bigcup \dots \bigcup B_n$

For any set $C \in \mathbf{Q}^n$ we denote by χ_C the characteristic function of C

The orthogonal projection

We denote by $f_C^{(h)}$ the orthogonal projection of the characteristic function χ_C onto the eigensubspace V_h. Then
$\chi_C = f_C^{(0)} + f_C^{(1)} + \ldots + f_C^{(n)}$.

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Fourier transform

$m_{\mathbf{xy}} = f^{\mathbf{x}}(\mathbf{y}) = (-1)^{\langle \mathbf{x},\mathbf{y} \rangle}, \ \mathbf{x}, \mathbf{y} \in \mathbf{Q}^n,$
The matrix $M = (m_{\mathbf{xy}})$ defines the orthogonal transform that is called Fourier transform.

Let $n = 2^t - 1$

Perfect codes: definition

The code $C \subseteq \mathbf{Q}^n$ is perfect if the balls of radius 1 centered in the code words do not intersect and cover all $\mathbf{Q}^n$.

The code distance is equal to 3.
Perfect codes of length n exist for every n of form $n = 2^t - 1$ and do not exist for any other n.

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The characteristic function

$\chi_C = f_C^{(0)} + f_C^{((n+1)/2)}$

It is easy to see that $f_C^{(0)} = \frac{1}{n+1}$.

Let $n = 4^t - 1$

Preparata codes: definition

A code is Preparata-like (or shortly, Preparata code), if
the length is $n = 4^t - 1$
the code distance is 5
the cardinality $2^{n+1}/(n+1)^2$.

F. P. Preparata. A class of optimum nonlinear double-error correcting codes. *Inf. Control*, 13(4):378–400, 1968.

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The characteristic function

$$\chi_P = f_P^{(0)} + f_P^{((n+1)/2)} + f_P^{(k)} + f_P^{(h)} \quad ,$$

$k = \frac{n+1}{2} - \frac{\sqrt{n+1}}{2}, \quad h = \frac{n+1}{2} + \frac{\sqrt{n+1}}{2}$
It is easy to see that $f_P^{(0)} = \frac{2}{(n+1)^2}$.

$P \subseteq C(P)$

Every Preparata code P is contained in a unique perfect code $C(P)$.

G. V. Zaitsev, V. A. Zinoviev, and N. V. Semakov. Interrelation of Preparata and Hamming codes and extension of Hamming codes to new double-error-correcting codes. In P. N. Petrov and F. Csaki, editors, *Proc. 2nd Int. Symp. Information Theory, Tsahkadsor, Armenia, USSR, 1971*, p. 257–264, Budapest, Hungary, 1973. Akademiai Kiado.

Perfect colorings: definition

A vertex partition $(T_0, \ldots, T_r)$ is called a perfect coloring (or equitable partition, or regular partition, or partition design) if for every $i, j \in \{0, \ldots, r\}$ there exists an integer s_{ij} such that every vertex from T_i has exactly s_{ij} neighbors from T_j.

The matrix $S = (s_{ij})$ is called the parameter matrix (or quotient) of the coloring.

Perfect colorings from Preparata codes

A Preparata code P induces a perfect coloring T by distances:
$T_0 = P, T_1, T_2, T_3 = C(P) \setminus P$

The parameter matrix: $S = \begin{bmatrix} 0 & n & 0 & 0 \\ 1 & 0 & n-1 & 0 \\ 0 & 2 & n-3 & 1 \\ 0 & 0 & n & 0 \end{bmatrix}$

with the eigenvalues $n, -1, -1 \pm \sqrt{n+1}$

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The characteristic function of the color T_i

$\chi_{T_i} = f_{T_i}^{(0)} + f_{T_i}^{(k)} + f_{T_i}^{((n+1)/2)} + f_{T_i}^{(h)}$

$k = \frac{n+1}{2} - \frac{\sqrt{n+1}}{2}$, $h = \frac{n+1}{2} + \frac{\sqrt{n+1}}{2}$.

For a Preparata code P we consider the function

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with the values

$$\begin{cases} \frac{n-1}{n+1}, & \mathbf{x} \in P \\ -\frac{2}{n+1}, & \mathbf{x} \in C(P) \setminus P \\ 0, & \mathbf{x} \notin C(P) \end{cases}$$

This function is antipodal.

Theorem. (V.)

Let P be an arbitrary Preparata code and $C(P)$ be the perfect code which contains P. Then

$$f_P^{((n+1)/2)} = \frac{2}{n+1} f_{C(P)}^{((n+1)/2)}.$$

Corollary. (V.)

Let P be a Preparata code. Then

$$\chi_P - \frac{2}{n+1}\chi_{C(P)} \in V_k \times V_h,$$

where $k = \frac{n+1}{2} - \frac{\sqrt{n+1}}{2}, \quad h = \frac{n+1}{2} + \frac{\sqrt{n+1}}{2}.$

Thank you for your attention!