ABOUT GROMOV'S THEOREM OF HOMOGENEOUS NILPOTENT APPROXIMATION

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On some domain $U \subset \mathbb{R}^N$ we consider basis vector fields $\{X_i\}_{i=1,\dots,N} \in C^1(U)$, i. e. $\operatorname{rank}\langle X_1,\dots,X_N\rangle(x) = N \quad \forall x \in U$, $\sup\limits_{x\in U}\|X(x)\| < C_U = \text{const}$. Fix a point $g \in U$ and consider maps $\theta_g : (a_1,\dots,a_N) \to \exp(X_a)(g)$, where $a = (a_1,\dots,a_N)$, $X_a = \sum\limits_{i=1}^{N} a_i X_i$, and $\vartheta_g^i : (a_1,\dots,a_N) \to \exp(X_{i_N}) \circ \dots \exp(X_{i_1})(g)$, $(i_1,\dots i_N) = \pi_i(1,\dots,N)$, where π_i is some permutation of the set $(1,\dots,N)$, $\exp(Y)(g)$ is the end-point of the integral line $\exp(tY)(g)$, $t \in [0,1]$ of some vector field Y, starting from point g. From well-known theorems of o. d. e. it follows, see, for example, [1], that for every point $g \in U$ there exists some neighbourhood of origin $O_g \subset \mathbb{R}^N$ such that the maps θ_g, ϑ_g^i are diffeomorphisms on O_g. Maps θ_g, ϑ_g^i are called canonical coordinates of the first and of the second types respectively, see, for example, [2]; map θ_g also is called exponential map. Let the symbol ϑ_g denotes the coordinate map of the second type inducing by the permutation $\pi(1,\dots,N) = (N, N-1,\dots,1)$. Suppose that the basis vector fields $\{X_i\}_{i=1,\dots,N} \in C^r(U)$, $r \geq 1$, which are formal graded by degrees, i. e. with every vector field X_i we associated some nonnegative integer $\deg X_i$ belonging to the set $\{1,\dots,N\}$, $\Upsilon = \max\limits_{i=1,\dots,N} \deg X_i$, satisfy in U the following table of commutators $[X_i, X_j] = \sum\limits_{\deg X_k \leq \deg X_i + \deg X_j} C_{ij}^k X_k$, $C_{ij}^k \in C^{r-1}(U)$. Then we say that basis vector fields $\{X_i\}_{i=1,\dots,N}$ satisfy in U $(+\deg)$ condition. In the first part of 90s M. Gromov in his famous work [3] published the following Theorem 1 about nilpotent approximation of C^1-smooth basis vector fields satisfying $(+\deg)$ condition.

Theorem 1. *In some neighbourhood of origin $O_g \subset \mathbb{R}^N$ the following uniform convergences take place:* $(\delta_{1/\varepsilon})_* \varepsilon^{\deg X_i} (\vartheta_g^{-1})_* X_i \to_{\varepsilon\to 0} \widehat{X}_i^g$, $i = 1,\dots,N$. *Vector fields $\{\widehat{X}_i^g\}_{i=1,\dots,N}$ satisfy the following table of commutators*

$$[\widehat{X}_i^g, \widehat{X}_j^g] = \sum_{\deg X_i + \deg X_j = \deg X_k} \widehat{C}_{ij}^k \widehat{X}_k^g, \quad \widehat{C}_{ij}^k = C_{ij}^k(g) = \text{const},$$

in O_g and form a basis of some Lie algebra L_g with structure constants $\widehat{C}_{ij}^k$.

Here $\delta_{1/\varepsilon}(x_1,\dots,x_N) = (\varepsilon^{-\deg X_i} x_1,\dots,\varepsilon^{-\deg X_N} x_N)$. Vector fields $\{(\theta_g)_* \widehat{X}_i^g\}_{i=1,\dots,N}$ are called homogeneous nilpotent approximation of vector fields $\{X_i\}_{i=1,\dots,N}$ in a neighbourhood of g, see [2]. Homogeneous nilpotent approximation take important place in theory of subelliptic equations [4], in problems of optimal control, etc. Note that methods of constructions of homogeneous nilpotent approximation in C^∞-case were developed quietly well in 70–90 years of the 20th century, see [2, 5]. Attention to Gromov's theorem of homogeneous nilpotent approximation were essentially inspired of problems of non-holonomic geometry under minimal conditions on smoothness of vector fields in the beginning of 2000s, see, for example, [6, 7]. Scrupulous considerations of the scheme of the proof of the Theorem 1 illuminated some its gaps. In particular in M. B. Karmanova and S. K. Vodopyanov's paper [7] it was given the example of the map $\vartheta_g \in C^\infty(\mathbb{R}^3)$ such that some properties, which were based for Gromov's proof, didn't take place. Later theorem of homogeneous nilpotent approximation for a basis vector fields, satisfying $(+\deg)$ condition, was proved for canonical C^1-vector fields (and as the

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consequence for general C^2-vector fields) [8], and for general $C^{1,\alpha}$-vector fields [9] with help to the methods which were different from method of [3].

Using the methods from [8], based on classical proof of the second Lie theorem [10], we established the following

Theorem 2 about nilpotent approximation C^1-smooth basis vector fields, satisfying $(+\deg)$ condition. *In some neibourhood of origin $O_g \subset \mathbb{R}^N$ the following uniform convergence* : $\delta_{1/\varepsilon}\varepsilon^{\deg X_i}(\theta_g^{-1})_* X_i \to_{\varepsilon\to 0} \widehat{X}_i^g$, $i = 1, \dots, N$, *take place. Vector fields* $\{\widehat{X}_i^g\}_{i=1,\dots,N}$ *in* O_g *satisfy the following table of commutators*

$$[\widehat{X}_i^g, \widehat{X}_j^g] = \sum_{\deg X_i + \deg X_j = \deg X_k} \widehat{C}_{ij}^k \widehat{X}_k^g, \quad \widehat{C}_{ij}^k = C_{ij}^k(g) = \mathrm{const},$$

and form a basis of some Lie algebra L_g *with structure constants* $\widehat{C}_{ij}^k$.

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