MORSE-SARD THEOREM FOR SOBOLEV FUNCTIONS AND APPLICATIONS

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Theorem 1 [1]–[2]. Let \( \psi \in W^{n,1}(\mathbb{R}^n) \). Then
(i) for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any set \( U \subset \mathbb{R}^n \) with \( H^1_\infty(U) < \delta \) the inequality \( H^1(\psi(U)) < \varepsilon \) holds;
(ii) \( H^1(\{\psi(x) : x \in \mathbb{R}^n & \nabla \psi(x) = 0\}) = 0 \).

Here we denote by \( H^1 \) the one-dimensional Hausdorff measure, i.e., \( H^1(F) = \lim_{t \to 0^+} H^1_t(F) \), where \( H^1_t(F) = \inf \{ \sum_{i=1}^{\infty} \text{diam} F_i : \text{diam} F_i \leq t, F \subset \bigcup_{i=1}^{\infty} F_i \} \).

Corollary 2 [1]–[2]. Let \( \psi \in W^{n,1}(\mathbb{R}^n) \). Then for \( H^1 \)-almost all \( y \in \psi(\mathbb{R}^n) \subset \mathbb{R} \) the preimage \( \psi^{-1}(y) \) is a finite disjoint family of \( C^1 \)-smooth \( (n-1) \)-dimensional compact manifolds \( S_j, j = 1, 2, \ldots, N(y) \).

Now consider the Euler system
\[
\begin{cases}
(w \cdot \nabla)w + \nabla p &= 0, \\
\text{div } w &= 0.
\end{cases}
\]

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with Lipschitz boundary. Assume that \( w = (w_1, w_2) \in W^{1,2}(\Omega, \mathbb{R}^2) \) and \( p \in W^{1,s}(\Omega), s \in [1,2], \) satisfy the Euler equations (1) for almost all \( x \in \Omega \) and let \( \int_{\Gamma_i} w \cdot n \, dS = 0, i = 1, 2, \ldots, N, \) where \( \Gamma_i \) are connected components of the boundary \( \partial \Omega \). Then there exists a stream function \( \psi \in W^{2,2}(\Omega) \) such that \( \nabla \psi = (-w_2, w_1) \) (note that by Sobolev Embedding Theorem \( \psi \) is continuous in \( \overline{\Omega} \) ). Denote by \( \Phi = p + \frac{|w|^2}{2} \) the total head pressure corresponding to the solution \( (w,p) \).

Theorem 3 [3] (Bernoulli Law for Sobolev solutions). Under above conditions, for any connected set \( K \subset \overline{\Omega} \) such that \( \psi|_K = \text{const} \) the assertion
\[
\exists C = C(K) \quad \Phi(x) = C \quad \text{for } H^1 \text{-almost all } x \in K
\]
holds.

Using Theorem 3 we prove the existence of the solutions to steady Navier–Stokes equations for some plane cases (see [4]) and for the spatial case when the flow has an axis of symmetry (see [5]).

REFERENCES

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