Modular standard modules of association schemes

Akihide Hanaki

We are trying to establish theory of modular representations of association schemes. Modular representations mean representations of adjacency algebras of association schemes over a positive characteristic field. In general, if two association schemes are algebraically isomorphic, namely they have the same intersection numbers, then the adjacency algebras over an arbitrary coefficient ring are isomorphic and representations are same. However, an association scheme have the special representation, the standard representation. Let (X, S) be an association scheme, and let R be a commutative ring with 1. We denote by $M_X(R)$ the full matrix algebra over R, rows and columns of whose matrices are indexed by the set X. The adjacency algebra RS is defined as a subalgebra of $M_X(R)$. Thus the inclusion $RS \to M_X(R)$ is a representation of (X, S), and we call this the *standard representation* over R.

In some papers, for example in [1, 5, 6], p-ranks of some integral matrices were considered and sometimes they could distinguish combinatorial objects with same parameters. In [4], it was pointed out that p-ranks were "shadows" of structures of standard modules.

In this talk, we will consider an extreme case, the case that the standard module is indecomposable. Let F be a field of positive characteristic p, and let G be a finite group. Then it is well known that the following conditions are equivalent :

(1) The group G is a p-group.
(2) The group algebra FG is a local algebra.
(3) The regular FG-module is indecomposable.

For an association scheme (X, S), it is known that

- If (X, S) is a p-scheme, then the adjacency algebra FS is a local algebra [2] and the standard module FX is an indecomposable FS-module [3].

By examples, we know that the conditions are not equivalent for association schemes. However, for schurian schemes, we have the following result.

Theorem 1. *Let F be a field of positive characteristic p, and let (X, S) be a schurian association scheme. Then (X, S) is a p-scheme if and only if the standard module FX is an indecomposable FS-module.*

A schurian association scheme is defined by a transitive permutation group, but the group is not unique, in general. It is easy to see that a transitive permutation p-group defines a schurian p-scheme. As a corollary to Theorem 1, we can show that the converse is also true.

Corollary 2. *Let G be a transitive permutation group. If G defines a schurian p-scheme, then G is a p-group.*

References

[1] A. E. Brouwer and C. A. van Eijl, *On the p-rank of the adjacency matrices of strongly regular graphs*, J. Algebraic Combin. **1** (1992), no. 4, 329–346.

[2] A. Hanaki, *Locality of a modular adjacency algebra of an association scheme of prime power order*, Arch. Math. (Basel) **79** (2002), no. 3, 167–170.

[3] A. Hanaki and O Shimabukuro, *Indecomposable decompositions of modular standard modules for two families of association schemes*, to appear in J. Algebraic Combin.

[4] A. Hanaki and M. Yoshikawa, *On modular standard modules of association schemes*, J. Algebraic Combin. **21** (2005), no. 3, 269–279.

[5] R. Peeters, *Uniqueness of strongly regular graphs having minimal p-rank*, Linear Algebra Appl. **226/228** (1995), 9–31.

[6] ———, *On the p-ranks of the adjacency matrices of distance-regular graphs*, J. Algebraic Combin. **15** (2002), no. 2, 127–149.

Faculty of Science, Shinshu University, Japan
E-mail address: `hanaki@shinshu-u.ac.jp`