ON COMMON ZEROES
OF THE LAPLACE – BELTRAMI EIGENFUNCTIONS

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Let $M$ be a compact connected closed orientable Riemannian $C^\infty$-manifold, $\Delta$ be the Laplace–Beltrami operator on it, and

$$E_\lambda = \{ u \in C^2(M) : \Delta u + \lambda u = 0 \}.$$

be the eigenspace for an eigenvalue $-\lambda > 0$ (we assume that the functions are real valued). Let $H^p(M)$ denote de Rham cohomologies.

**Theorem 1** ([1]). Let $M$ be as above.

1. If $H^1(M) = 0$, then for any $\lambda \neq 0$ and every $u, v \in E_\lambda$ there exists $p \in M$ such that $u(p) = v(p) = 0$.

2. If $M$ is a homogeneous space of a compact Lie group $G$ acting by isometries, then the converse is true: $H^1(M) \neq 0$ implies the existence of $\lambda \neq 0$ and a pair of $u, v \in E_\lambda$ without common zeroes.

There is a simple example for the second assertion: let $G$ be the circle group $T = T(2\pi \mathbb{Z})$, which acts on itself by the translations. Then $u(t) = \cos t$, $v(t) = \sin t$ have no common zero. In fact, (2) follows from this example and the existence of $G$-equivariant mapping $M \rightarrow T$ for a nontrivial action of $G$ on $T$, which is a consequence of the assumption $H^1(M) \neq 0$.

For an eigenfunction $u$, $N_u = u^{-1}(0)$ is said to be the nodal set, and connected components of $M \setminus N_u$ are called nodal domains. The proof of (1) is based on the following properties of them:

A) if $U, V$ are nodal domains for $u, v \in E_\lambda$, respectively, and $V \subseteq U$, then $u = cv$ for some $c \in \mathbb{R}$;

B) $u \in E_\lambda \setminus \{0\}$ cannot keep its sign near every point of $N_u$.

For a homogeneous space $M = G/H$ and any $G$-invariant Riemannian metric on $M$, each $G$-irreducible invariant subspace $E \subseteq L^2(M, \sigma)$, where $\sigma$ is the invariant measure with the total mass 1 on $M$, is contained in some $E_\lambda$. For $a \in M$, let $\phi_a \in E$ be the function which realizes the evaluation functional at $a$: $\langle u, \phi_a \rangle = u(a)$ for all $u \in E$. For $a_1, \ldots, a_k, x, y \in M$, set $a = (a_1, \ldots, a_k) \in M^k$, $\phi(x, y) = \langle \phi_x, \phi_y \rangle$, and

$$\Phi_{k,y}^a(x) = \det \begin{pmatrix} \phi(a_1, a_1) & \ldots & \phi(a_1, a_k) & \phi(a_1, y) \\ \vdots & \ddots & \vdots & \vdots \\ \phi(a_k, a_1) & \ldots & \phi(a_k, a_k) & \phi(a_k, y) \\ \phi(x, a_1) & \ldots & \phi(x, a_k) & \phi(x, y) \end{pmatrix}.$$

Further, let $\Phi_k$ be the mapping $(a, y) \rightarrow \Phi_{k,y}^a$, $n = \dim E - 1$, and set $U_n = \Phi_n(M^{n+1})$.

**Theorem 2.** If $u \in E$, $u \neq 0$, then there exists a nontrivial continuous function $c(a, y)$ on $(N_u)^n \times M$ such that $\Phi_{n,y}^a = c(a, y)u$. Moreover, $U_n$ is a compact symmetric neighbourhood of zero in $E$. For every $a \in M^n$, there exists a nontrivial nodal set which contains $a$; for a generic $a$, this set is unique.

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The construction, which is classical, can be applied to each finite dimensional space of continuous functions but usually the set $U_n$ is small. The proof essentially uses (A).

Let $M$ be the unit sphere $S^2 \subset \mathbb{R}^3$. Then $\lambda_n = n(n+1)$ is the $n$-th eigenvalue. The corresponding eigenspace $E_n = E_{\lambda_n}$ consists of spherical harmonics; they can be defined as restrictions to $S^2$ of homogeneous polynomials of degree $n$ on $\mathbb{R}^3$ which are harmonic with respect to the ordinary Laplacian in $\mathbb{R}^3$. The zonal spherical functions $\phi_a$ can be written explicitly by the $n$-th Legendre polynomial; $\dim E_n = 2n+1$. Let $\nu(u, v)$ be the number of points in $N_u \cap N_v$, where $u, v \in E_n$. This set can be infinite. However, $\nu(u, v) \leq 2n^2$ for generic $u, v \in E_n$, and there are examples of $u, v$ such that $\nu(u, v) = 2n^2$. The greatest lower bound is not known but partial results and computer experiments support the following conjecture: $\nu(u, v) \geq 2n$ for all $u, v \in E_n$. Also, there are examples for the equality (for all $n > 0$).

For problems and results (up to 2001) on the geometry of eigenfunctions, see the survey [2] and references in it.

REFERENCES