EFFECTIVE INSEPARABILITY AND POSITIVE STRUCTURES

ANDREA SORBI

We examine the notion of effective inseparability, and how it has been recently exploited in the study of positive structures, in particular positive lattices [2].

A positive lattice L is said to be *uniformly effectively inseparable* (abbreviated as *u.e.i.*) if its equality relation corresponds to a positive equivalence relation $=_L$ yielding a partition of the set ω of natural numbers in equivalence classes which are pairwise effectively inseparable in a uniform way (for applications of uniform effective inseparability to positive equivalence relations see [1]). The following hold of a u.e.i. positive lattice L, and the preordering relation $\leq_L$ on ω corresponding to the partial ordering relation of the lattice: (1) $\leq_L$ is *locally universal* for the class of positive preordering relations, i.e. inside any nonempty interval of $\leq_L$ one can computably embed any positive preordering relation; (2) $\leq_L$ is *uniformly dense*, i.e. there exists a computable function f such that if $x <_L y$ then $x <_L f(x, y) <_L y$, and f is well defined on equivalence classes of $=_L$: if $x =_L x'$ and $y =_L y'$ then $f(x, y) =_L f(x', y')$.

Luckily, to check that a given nontrivial positive bounded lattice L is u.e.i., it is enough to check effective inseparability of the pair $(0_L, 1_L)$ of equivalence classes corresponding to the least element and the greatest element, respectively, of the lattice. This has obvious applications to the study of formal systems, and Lindenbaum lattices of sentences. A *Lindenbaum lattice of sentences* (for this topic see [5]) is a positive bounded lattice $L_{\mathcal{C},T}$, where T is a recursively enumerable (r.e.) consistent theory, such that: its universe is given by an r.e. set of sentences $\mathcal{C}$ which is closed under the connectives $\wedge$ and $\vee$; its preordering relation is given by provable implication in T; lattice equality is given by provable equivalence in T; the sentences of $\mathcal{C}$ which are refutable in T correspond to the least element of the lattice, and the sentences of $\mathcal{C}$ which are provable in T correspond to the greatest element. In these cases, in order to show that the preordering relation of $L_{\mathcal{C},T}$ is locally universal and uniformly dense, it is enough to prove effective inseparability of the pair of sets $(0_{L_{\mathcal{C},T}}, 1_{L_{\mathcal{C},T}})$. Examples not previously noticed in the literature include Lindenbaum lattices $L_{\mathcal{C},T}$ where: (a) T is any r.e. consistent extension of either one of Robinson's systems R or Q, and $\mathcal{C}$ contains the $\exists\Delta_0$ sentences; the same holds if we let T be any r.e. consistent intuitionistic extension of either one of the intuitionistic versions of Robinson's systems; (b) T is any r.e. consistent extension of Buss' weak arithmetical system S^1_2, and $\mathcal{C}$ is the class $\exists\Sigma^{\mathsf{b}}_1$ of sentences.

We examine also applications of effective inseparability ([4, 3]) to the general problem of which positive equivalence relations can be realized (up to computable isomorphism, or computable bi-embeddability of equivalence relations) as word problems of positive algebras.

References

[1] U. Andrews, S. Lempp, J. S. Miller, K. M. Ng, L. San Mauro, and A. Sorbi, *Universal computably enumerable equivalence relations*, J. Symbolic Logic **79** (2014), no. 1, 60–88.
[2] U. Andrews and A. Sorbi, *Effective inseparability, lattices, and pre-ordering relations*, Review of Symbolic Logic (2019), 1–28, DOI 10.1017/S1755020319000273.
[3] V. Delle Rose, L. F. San Mauro, and A. Sorbi, *Word problems and ceers*, LQ Math. Log. Q. **66** (2020), no. 3, 341–354.
[4] A. Nies and A. Sorbi, *Calibrating word problems of groups via the complexity of equivalence relations*, Mathematical Structures in Computer Science (2018), 1–15.
[5] V. Yu. Shavrukov and Albert Visser, *Uniform density in Lindenbaum algebras*, Notre Dame J. Form. Log. **55** (2014), no. 4, 569–582.

Department of Information Engineering and Mathematics, University of Siena, 53100 Siena, Italy
E-mail address: andrea.sorbi@unisi.it