GRÖBNER—SHIRSHOV BASES FOR CONFORMAL AND VERTEX ALGEBRAS

P. S. KOLESNIKOV, R. A. KOZLOV

The theory of vertex algebras (or vertex operator algebras, VOAs) appeared in mathematical physics as an algebraic tool for studying the operator product expansion (OPE) of chiral fields in 2-dimensional conformal field theory which goes back to A. Belavin, A. Polyakov, and A. Zamolodchikov (1984). The algebraic definition of a VOA was first stated by R. Borcherds (1986). The development of the theory of vertex algebras is mainly carried out within the framework of the representation theory. In order to define a vertex algebra in this way, one need to get the base space V, a linear operator $T : V \to V$ (translation), a selected vector $\mathbf{1} \in V$ (vacuum) such that $T\mathbf{1} = 0$, and define a family of vertex operators, formal distributions $Y(a, z) \in gl(V)[[z, z^{-1}]]$, $a \in V$, satisfying certain properties. The Dong Lemma along with the Goddard Uniqueness Theorem show that it is enough to define the series $Y(a, z)$ not for all $a \in V$, but just for "generators".

We will consider a combinatorial approach to the construction of a vertex algebra. In order to get a normal form of a vertex algebra defined by generators and relations one need to solve a Gröbner—Shirshov basis problem for a module over an appropriate associative algebra. Occasionally, the same approach works well for associative conformal algebras.

Let Vert, LSym, and LieConf be the categories of vertex, pre-Lie, and Lie conformal algebras, respectively. As follows from the definition, there are two forgetful functors

$$\Phi : \mathrm{Vert} \to \mathrm{LieConf}, \quad \Psi : \mathrm{Vert} \to \mathrm{LSym}.$$

The first one was studied by M. Roitman (2000), where the left adjoint functor for Φ was explicitly constructed. Every Lie conformal algebra L embeds into its universal enveloping vertex algebra $V(L)$, and there is an analogue of the Poincaré–Birkhoff–Witt (PBW) Theorem on the linear basis of $V(L)$.

The second functor Ψ has completely different properties. We show that there exist pre-Lie (super)algebras that cannot be embedded into a vertex (super)algebra. Namely, let A be a pre-Lie (super)algebra. If A embeds into a vertex (super)algebra in such a way that $a.b = ab$ for all $a, b \in A$ and the locality function on A is bounded then the commutator Lie (super)algebra $A^{(-)}$ is nilpotent. In particular, as follows from the results of [11], a finite-dimensional simple pre-Lie algebra cannot be embedded into a vertex algebra.

This research is supported by Russian Science Foundation (project 21-11-00286).

Sobolev institute of mathematics, Novosibirsk (Russia)
E-mail address: `pavelsk@math.nsc.ru`