Discretization of theory of Riemann surfaces and graphs of groups

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Introduction

A *graph* X is

- the set $V(X)$ of vertices;
- the set $E(X)$ of edges;
- maps $s, t : E(X) \to V(X)$ (endpoints);
- a fixed point free involution $e \to \bar{e}$ of $E(X)$ such that
  $s(\bar{e}) = t(e)$ and
  $t(\bar{e}) = s(e)$.

We work with finite connected multigraphs without loops.

The *genus* of a graph is defined to be the rank of the first homology group of the graph (that is, its cyclomatic number).

Harmonic morphism of graphs

- A *morphism* of graphs $f : X \to Y$ sends vertices to vertices, edges to edges, and preserve the source, target and reverse of each $e \in E(X)$.
- We call a morphism f a *covering*, if it is surjective and locally bijective.

A non-constant holomorphic map $M \to N$ between Riemann surfaces locally pulls back a holomorphic function on N to a holomorphic function on M.
Analogue for graphs – *harmonic* morphisms and harmonic functions on graphs.

A morphism of graphs $f : X \to Y$ is said to be *harmonic* or *branch covering*, if, for each $x \in V(X)$, $y \in V(Y)$ such that $y = f(x)$, the quantity $|e \in E(X) : s(e) = x, f(e) = e'|$ is the same for all $e' \in E(Y)$, such that $s(e') = y$.

A composition of harmonic morphisms is harmonic.
A covering of graphs is harmonic morphism.

Harmonic morphism of graphs

Let $f : X \to Y$ be a harmonic morphism.

For any $x \in V(X)$ we define a *multiplicity* of f at x by

$$m_f(x) = |e \in E(X) \, : \, s(e) = x, f(e) = e'|$$

for any $e' \in E(Y)$ such that $s(e') = f(x)$.

A *degree* of f is defined by

$$\deg(f) = |e \in E(X) \, : \, f(e) = e'|$$

for any $e' \in E(Y)$.

Uniformization of Riemann surfaces

Let M and N be compact Riemann surfaces of genus greater than 1, and $M \to N$ be a non-constant holomorphic map of degree n.

Let D stand for the unit disk, and $Aut(D)$ be the group of all its conformal automorphisms.

Then there exist groups $\Gamma < Aut(D)$ and $H < Aut(D)$, such that $M \cong D/H$, $N \cong D/\Gamma$, $H < \Gamma$ and $[\Gamma : H] = n$. Moreover, if $H \lhd \Gamma$, then $N \cong M/(\Gamma/H)$.

To organize similar uniformization for harmonic morphisms of graphs, we apply Bass-Serre theory of graph of groups.

Graphs of groups

A *graph of groups* $\mathbb{X} = (X, \mathcal{A})$ is

1. a graph X;
2. for any $a \in V(X)$, a group $\mathcal{A}_a$;
   for any $e \in E(X)$, a group $\mathcal{A}_e = \mathcal{A}_{\bar{e}}$;
3. monomorphisms $\alpha_e : \mathcal{A}_e \to \mathcal{A}_a$, where $a = s(e)$.

Let $\mathbb{X} = (X, \mathcal{A})$ be a graph of groups.
Choose a spanning tree T in X. Then the fundamental group of $\mathbb{X}$ relative to T is

$$\pi_1(\mathbb{X}, T) = \left[\left(\mathop{*}_{a \in V(X)} \mathcal{A}_a\right) * F(E(X))\right] / R, \quad \text{where}$$

F(E(X)) is a free group on $E(X)$;
R – relations:

- $e\alpha_{\bar{e}}(g)\bar{e} = \alpha_e(g)$ for all $e \in E(X)$ and $g \in \mathcal{A}_e$;
- $\bar{e} = e^{-1}$ for all $e \in E(X)$;
- $e = 1$ for all $e \in E(T)$.

Covering of graphs of group

In an application of Bass-Serre theory to harmonic morphisms of graphs, we need only graphs of groups with trivial group assigned to each edge of an underlying graph.

Let $\mathbb{X} = (X, \mathcal{A})$ and $\mathbb{Y} = (Y, \mathcal{B})$ be graphs of groups with trivial group assigned to each edge of X and Y.

We define a covering of graphs of groups $\mathbb{F} = (f, \Phi) : \mathbb{X} \to \mathbb{Y}$ to consist of

- a harmonic morphism $f : X \to Y$;
- a set Φ of monomorphisms $\phi_a : \mathcal{A}_a \to \mathcal{B}_{f(a)}$ $(a \in V(X))$ such that $m_f(a)|\mathcal{A}_a| = |\mathcal{B}_{f(a)}|$, where $m_f(a)$ – the multiplicity of f at a.

Uniformization of graphs of group

For any graph of groups $\mathbb{X}$, there exists the universal covering infinite tree $\widetilde{\mathbb{X}}$, on which the fundamental group $\pi_1(\mathbb{X})$ acts without invertible edges and $\widetilde{\mathbb{X}}/\pi_1(\mathbb{X}) \cong \mathbb{X}$.

Bass uniformization theorem

Let $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ be a covering of graphs of groups and $\mathbb{F}_{\pi_1} : \pi_1(\mathbb{X}) \to \pi_1(\mathbb{Y})$ the induced homomorphism of the fundamental groups. Then there is a lift of $\mathbb{F}$ to a $\mathbb{F}_{\pi_1}$-equivariant isomorphism $\widetilde{\mathbb{F}} : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{Y}}$ between the universal covering trees.

$$\begin{tikzcd}
\widetilde{\mathbb{X}} \arrow[r, "\widetilde{\mathbb{F}}"] \arrow[d, "p"] & \widetilde{\mathbb{Y}} \arrow[d, "p'"] \\
\mathbb{X} \arrow[r, "\mathbb{F}"] & \mathbb{Y}
\end{tikzcd}$$

Let $H = \pi_1(\mathbb{X})$ and $\Gamma = \pi_1(\mathbb{Y})$.
Recall that $\widetilde{\mathbb{X}}/H \cong \mathbb{X}$ and $\widetilde{\mathbb{Y}}/\Gamma \cong \mathbb{Y}$.
By the theorem, instead of $\mathbb{F} : \mathbb{X} \to \mathbb{Y}$ we can work with $\mathbb{F}' : \widetilde{\mathbb{X}}/H \to \widetilde{\mathbb{Y}}/\Gamma$ induced by the group inclusion $H < \Gamma$ and $[\Gamma : H] = \deg(f)$.
According to Green's PhD, if $H \lhd \Gamma$ then $\mathbb{Y} \cong \mathbb{X}/(\Gamma/H)$.

Groups with partition

A finite group G is said to admit a *partition* if it can be expressed as a set-theoretic union of subgroups, with pairwise trivial intersections.

Given a compact Riemann surface M with automorphism group G_0, we can obtain a quotient surfaces M/G_i, where G_i are subgroups of G_0. Accola derived the formula relating the genera of M/G_i to the orders of G_i provided G_0 admits a partition. Taniguchi generalized this result to finite groups acting on a compact Hausdorff space.

Corollary of Taniguchi's theorem

Let X be a graph on which a finite group G_0 acts, and assume that G_0 admits a partition $\{G_1, G_2, \ldots, G_s\}$. Then we have

$$(s-1)g(X) + |G_0|g(X/G_0) = \sum_{i=1}^{s} |G_i|g(X/G_i).$$

Main result

A graph X is said to be γ-*hyperelliptic*, if there exists a degree 2 harmonic morphism $X \to Y$, where graph Y is of genus γ. In this case, there exists a involution $\tau \in Aut(X)$, acting freely on the set of edges of X and without invertible edges, such that $X/ < \tau > \cong Y$.

Theorem

Let X be a degree 2 covering of a hyperelliptic graph Y of genus $g \geq 2$. Then X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.

Proof:

1. We have $\varphi : X \to Y$ and $\psi : Y \to T$. The composite mapping $F = \varphi \circ \psi$ is a harmonic morphism.
2. Denote by $\mathbb{X}$ and $\mathbb{T}$ graph of groups. The map $F : X \to T$ can be naturally extended to the covering $\mathbb{F} : \mathbb{X} \to \mathbb{T}$ of graph of groups.
3. Let $\mathrm{H} = \pi_1(\mathbb{X})$ and $\Gamma = \pi_1(\mathbb{T})$ be the fundamental groups, and $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{T}}$ be the universal covering trees of graphs of groups $\mathbb{X}$ and $\mathbb{T}$ respectively.

Proof

4. By the Bass uniformization theorem there exists a lift of $\mathbb{F}$ to an isomorphism $\widetilde{\mathbb{F}} : \widetilde{\mathbb{X}} \to \widetilde{\mathbb{T}}$ between covering trees equivariant under the action of H and Γ on $\widetilde{\mathbb{X}}$ and $\widetilde{\mathbb{T}}$ respectively.
5. We have $\mathbb{X} \cong \widetilde{\mathbb{X}}/\mathrm{H}$ and $\mathbb{T} \cong \widetilde{\mathbb{T}}/\Gamma$.
6. Replace the covering $\mathbb{F} : \mathbb{X} \to \mathbb{T}$ by the covering $\mathbb{F}' : \widetilde{\mathbb{X}}/\mathrm{H} \to \widetilde{\mathbb{X}}/\Gamma$ induced by the group inclusion $H < \Gamma$.

Lemma

Let Γ be a free product of $n > 1$ copies of $\mathbb{Z}_2$. If $H < \Gamma$ is a torsion-free subgroup of index 4, then $H \lhd \Gamma$.

7. By Lemma, H is a normal subgroup of index 4 in Γ. Therefore, by Green's PhD, the covering transformation group of $\mathbb{F}'$ is $G_0 = \Gamma/H = V_4$.
8. V_4 admits a partition $\{G_1, G_2, G_3\}$ into three subgroups of order two.

Proof

9. Use $(s-1)g(X) + |G_0|g(X/G_0) = \sum_{i=1}^{s} |G_i|g(X/G_i)$.
   We get $g - 1 = g_1 + g_2$. The possible cases for g_1 and g_2 are

g_1	g_2
0	$g-1$
1	$g-2$
$\dots$	$\dots$
$\left[\dfrac{g-1}{2}\right]$	$\left[\dfrac{g-1}{2}\right]$ $(+1,$ if g is even).

   Choosing the smaller genus in each case, we get that X is γ-hyperelliptic for some $\gamma \leq \left[\frac{g-1}{2}\right]$.

Corollaries

The immediate consequences of the theorem are the assertions below. The first one has been proved by I. A. Mednykh by sophisticated methods.

Corollary 1

Suppose X is a graph of genus 3 which is a degree 2 covering of a graph Y of genus 2. Then X is hyperelliptic.

Corollary 2

If X is a graph of genus 5 which is a degree 2 covering of a hyperelliptic graph of genus 3, then X is hyperelliptic or 1-hyperelliptic.