Which isospectral graphs are isomorphic?

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Introduction

The theory of Riemann surfaces was founded in classical works by B. Riemann and A. Hurwitz. We note that originally Riemann surface was defined as a branched covering over the sphere. Over the last decade, a few discrete versions of the theory of Riemann surfaces were created.

1. Bacher, R., P. de la Harpe, and Nagnibeda,T., 1997
2. Urakawa, H., 2000
3. Baker, M., Norine, S., 2009
4. Caporaso, L., 2011
5. Corry, S., 2013

In these theories, the role of Riemann surfaces is played by graphs. As well as branched coverings are replaces by quasi-coverings of graphs.

Dictionary

1. Riemann surface $\Longleftrightarrow$ Finite connected multigraph
2. Holomorphic map $\Longleftrightarrow$ Harmonic map
   (branched covering) (quasi-covering)
3. The sphere $\Longleftrightarrow$ Tree
4. Torus (= one "hole" surface) $\Longleftrightarrow$ Flower (= one cycle graph)
5. Genus ($\sharp$ of "holes") $\Longleftrightarrow$ Genus ($\sharp$ of independent cycles)
6. Conformal automorphism $\Longleftrightarrow$ Automorphism acting free on arcs

Isospectral surfaces and graphs

Since the classical paper by Mark Kac "Can one hear the shape of a drum?"(1966), the question of what geometric properties of a manifold are determined by its Laplace operator has inspired many intriguing results.

Wolpert (1979) showed that a generic Riemann surface is determined by its Laplace spectrum. Nevertheless, pairs of isospectral non-isometric Riemann surfaces in every genus ≥ 4 are known. See papers by Buser (1986), Brooks and Tse (1987), and others. There are also examples of isospectral non-isometric surfaces of genus two and three with variable curvature Barden and Hyunsuk Kang (2012). In the same time, isospectral genus one Riemann surfaces (flat tori) are isometric (Brooks, 1988). Similar result for Klein bottle was obtained by R. Isangulov (2000). Similar results are also known for graphs (see survey by E.R.van Dam and W.H.Haemers (2003)).

Peter Buser (1992) posed an interesting problem: are two isospectral Riemann surfaces of genus two isometric? Up to our knowledge the problem is still open but, quite probably, can be solved positively. The aim of this paper is to give a positive solution of this problem for graphs of genus two. Because of the intrinsic link between Riemann surfaces and graphs we hope that our result will be helpful to make a progress in solution of the Buser problem.

The graphs in this paper are unoriented, but they may have loops and multiple edges. Denote by $V(G)$ and $E(G)$, respectively, the set of vertices and edges of a graph G. For each $u, v \in V(G)$, we set a_{uv} to be equal to the number of edges between u and v.
The matrix $A = A(G) = [a_{uv}]_{u,v \in V(G)}$, is called the *adjacency matrix* of the graph G. Let $d(v)$ denote the valency of $v \in V(G)$, $d(v) = \sum_u a_{uv}$, and let $D = D(G)$ be the diagonal matrix indexed by $V(G)$ and with $d_{vv} = d(v)$.
The matrix $L = L(G) = D(G) - A(G)$ is called the *Laplacian matrix* of G. It should be noted that loops have no influence on $L(G)$. Throughout the paper we shall denote by $\mu(G, x)$ the characteristic polynomial of $L(G)$. For brevity, we will call $\mu(G, x)$ the *Laplacian polynomial* of G.

Laplace operator on graphs

The roots of $\mu(G,x)$ will be called the Laplacian eigenvalues (or sometimes just eigenvalues) of G. They will be denoted by

$$\mu_1(G) \le \mu_2(G) \le \ldots \le \mu_n(G),\ (n = |V(G)|),$$

always enumerated in increasing order and repeated according to their multiplicity. Recall that for connected graph G we always have $\mu_1(G) = 0$ and $\mu_2(G) > 0$.

Two graphs G and H are called *Laplacian isospectral* (or *isospectral*) if their Laplacian polynomials coincide $\mu(G,x) = \mu(H,x)$.

Genus of graphs

By a graph we mean a finite connected multigraph, possibly with loops and multiple edges. We define genus of graph ? as

$$g = |E(G)| - |V(G)| + 1.$$

In graph theory, the term "genus"is traditionally used for a different concept, namely, the smallest genus of any surface in which the graph can be embedded, and the integer $g = g(G)$ is called the cyclomatic or the Betti number of G. We call g the genus of G in order to highlight the analogy with Riemann surfaces.
A *bridge* is an edge of a graph G whose deletion increases the number of connected components. A graph is said to be *bridgeless* if it contains no bridges.

A. K. Kel'mans (1967) gave a combinatorial interpretation to all the coefficients of $\mu(X, x)$ in terms of the numbers of certain subforests of the graph. We present the result in the following form.

Theorem

If $\mu(X, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$ *then*

$$c_i = \sum_{S \subset V,\, |S| = n-i} T(X_S),$$

where $T(H)$ *is the number of spanning trees of* H, *and* X_S *is obtained from* X *by identifying all points of* S *to a single point.*

Theta graphs

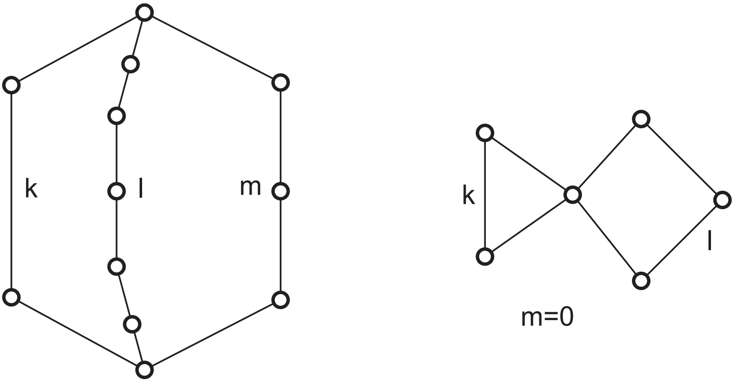


Fig.1. Theta graph $\Theta(k, l, m)$.

Theta graphs

Let u and v are two (not necessary distinct) vertices. Denote by $\Theta(k, l, m)$ the graph consisting of three internally disjoint paths joining u to v with lengths $k, l, m \geq 0$ (see Fig. 1). We set
$\sigma_1 = \sigma_1(k, l, m) = k + l + m, \sigma_2 = \sigma_2(k, l, m) = k\,l + l\,m + k\,m$, and
$\sigma_3 = \sigma_3(k, l, m) = k\,l\,m$.
It is easy to see that two graphs $\Theta(k, l, m)$ and $\Theta(k', l', m')$ are isomorphic if and only if the unordered triples $\{k, l, m\}$ and $\{k', l', m'\}$ coincide.
Equivalently, $\sigma_1 = \sigma_1', \sigma_2 = \sigma_2'$ and $\sigma_3 = \sigma_3'$, where
$\sigma_1' = \sigma_1(k', l', m'), \sigma_2' = \sigma_2(k', l', m')$, and $\sigma_3' = \sigma_1(k', l', m')$.

Lemma 1

Let G be an arbitrary bridgeless graph of genus two. Then G is isomorphic to $\Theta(k, l, m)$ for some k, l, m with $\sigma_2 = k\,l + l\,m + k\,m > 0$.

Proof. Since the graph G is bridgeless it has no vertices of valency one. Denote by H the graph obtained from G by deleting of all vertices of valency two. Suppose that H has V vertices of valences $n_1, n_2, \ldots, n_V$ and E edges. Since the valency of each vertex of H is at least three we have $n_i \geq 3, i = 1, 2, \ldots, V$. We have $g(H) = g(G) = 2$. Thus $g(H) = 1 - V + E = 2$ and $E = V + 1$. Counting the sum of valences of H through vertices and through edges we obtain

$$n_1 + n_2 + \ldots + n_V = 2E.$$

Hence $3V \leq n_1 + n_2 + \ldots + n_V = 2E = 2V + 2$, or $V \leq 2$. Then we have one of the two graphs shown on Fig.1.

The main result

The first result is the following theorem.

Theorem

Two genus two bridgeless graphs are Laplacian isospectral if and only if they are isomorphic.

The proof of the theorem is based on the following three lemmas.

Lemma 2.

Lemma 2

Let $G = \Theta(k, l, m)$ *be a theta graph and*

$$\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$$

is its Laplacian polynomial. Then $n = k + l + m - 1, c_1 = 2(k + l + m)$ *and* $c_{n-1} = (k\,l + l\,m + k\,m)(k + l + m - 1)$.

Proof. The number of vertices, edges and spanning trees of graph G are given by

$$V(G) = k + l + m - 1,\ E(G) = k + l + m,\ T(G) = k\,l + l\,m + k\,m.$$

Then by the Kel'mans theorem we have
$n = V(G) = k + l + m - 1,\ c_1 = 2E(G) = 2(k + l + m)$ and
$c_{n-1} = V(G) \cdot T(G) = (k\,l + l\,m + k\,m)(k + l + m - 1)$.

Lemma 3

Let $G = \Theta(k, l, m)$ be a theta graph and $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ is its Laplacian polynomial. Then

$$c_{n-2} = A(\sigma_1,\, \sigma_2) + B(\sigma_1,\, \sigma_2)\sigma_3$$

where $A(s, t) = (4t - 3st - 2s^2 t + s^3 t + 4t^2 - st^2)/12,\ B(s, t) = (3 - 4s + s^2 - 3t)/12,$
$\sigma_1 = k + l + m,\ \sigma_2 = k\,l + l\,m + k\,m,$ *and* $\sigma_3 = k\,l\,m.$

Proof. By the Kelmans theorem

$$c_{n-2} = \sum_{S \subset V,\, |S|=2} T(X_S),$$

where X_S runs through all graphs obtained from $G = \Theta(k, l, m)$ by gluing two vertices. There are exactly four types of such graphs shown on the Fig.2. We will enumerate the spanning trees of each type separately.

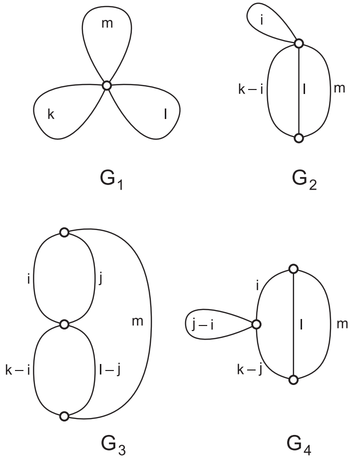


Fig. 2. The graphs obtained from $\Theta(k, l, m)$ by gluing two vertices

Lemma 3.

Type G_1. Glue two 3-valent vertices of graph G. As a result we obtain the graph G_1 shown on Fig. 2. The number of spanning trees of this graph is $T_1 = T(C_k) \cdot T(C_l) \cdot T(C_m) = k\, l\, m$.

Type G_2. Glue one 3-valent and one 2-valent vertices of graph G. For given i, $1 \leq i \leq k-1$ the number of spanning trees for graph G_2 is equal to $T(C_i) \cdot T(\Theta(k-i, l, m)) = i\sigma_2(k-i, l, m)$. We set $F(k, l, m) = \sum\limits_{i=1}^{k-1} i\sigma_2(k-i, l, m)$. Then the total number of spanning trees for graphs of type G_2 is

$$T_2 = 2(F(k, l, m) + F(l, m, k) + F(m, k, l)).$$

The multiple 2 is needed since the graph $\Theta(k, l, m)$ has two 3-valent vertices.

In a similar way we calculate the numbers T_3 and T_4.
Finally, we have

$$c_{n-2} = T_1 + T_2 + T_3 + T_4 = A(\sigma_1, \sigma_2) + B(\sigma_1, \sigma_2)\sigma_3.$$

Lemma 4

Let $G = \Theta(k, l, m)$ *be a theta graph and*
$\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ *is its Laplacian polynomial.*
Then

$$c_{n-3} = C(\sigma_1, \sigma_2) + D(\sigma_1, \sigma_2)\sigma_3 + E(\sigma_1, \sigma_2)\sigma_3^2,$$

where

$$\begin{aligned} C(s,t) &= (-34t + 21st + 25s^2t - 10s^3t - 3s^4t + s^5t - 50t^2 + 10st^2 \\ &+ 12s^2t^2 - 2s^3t^2 - 16t^3 + st^3)/360, \\ D(s,t) &= (-45 + 50s + 5s^2 - 12s^3 + 2s^4 + 24st - 9s^2t + 15t^2)/360, \\ E(s,t) &= -3(-8 + 3s)/360. \end{aligned}$$

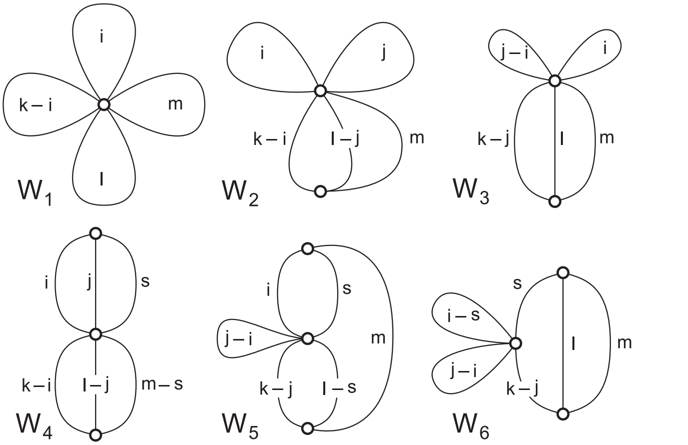


Fig. 3. The graphs obtained from $\Theta(k, l, m)$ by gluing three vertices

Lemma 4.

The proof of Lemma 4 is based on similar arguments given in the proof of Lemma 3. One can consider the six cases shown on the Fig. 3.

Proof of the Main Theorem

Let G and G' be two bridgeless graphs of genus two. Then by Lemma 1 for suitable $\{k, l, m\}$ and $\{k', l', m'\}$ we have

$$G = \Theta(k, l, m) \text{ and } G' = \Theta(k', l', m')$$

Denote by $\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$ and $\mu(G', x) = x^{n'} - c_1 x^{n'-1} + \ldots + (-1)^{n'-1} c_{n'-1} x$ their Laplacian polynomials.

Proof of the Main Theorem

Suppose that the graphs G and G' are isospectral. Then $n' = n, c'_1 = c_1, \ldots, c'_{n-1} = c_{n-1}$. By Lemma 2 we obtain

$$2\sigma_1 = 2\sigma'_1 \text{ and } \sigma_2(\sigma_1 - 1) = \sigma'_2(\sigma'_1 - 1).$$

Since both graphs are of genus 2 we have $\sigma_1 > 1$ and $\sigma'_1 > 1$. Then the obtained system of equations gives $\sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$. The theorem will be proved if we show that $\sigma_3 = \sigma'_3$. We will do this in two steps. First of all, we note that isospectral graphs with $n \leq 5$ vertices are isomorphic. So, we can assume that $n = k + l + m - 1 > 5$, that is $\sigma_1 = k + l + m > 6$.

By Lemma 3

$$c_{n-2} = A(\sigma_1, \sigma_2) + B(\sigma_1, \sigma_2)\sigma_3,$$

where $A(s,t) = (4t - 3st - 2s^2t + s^3t + 4t^2 - st^2)/12$ and $B(s,t) = (3 - 4s + s^2 - 3t)/12$.

Step 1. $B(\sigma_1, \sigma_2) \neq 0$. Since $c'_{n-2} = c_{n-2}, \sigma_1 = \sigma'_1$ and $\sigma_2 = \sigma'_2$ we obtain

$$B(\sigma_1, \sigma_2)\sigma'_3 = B(\sigma_1, \sigma_2)\sigma_3.$$

Hence $\sigma_3 = \sigma'_3$ and the theorem is proved.

Step 2. $B(\sigma_1,\sigma_2)=0$. Then $c'_{n-3}=c_{n-3}$ gives

$$D(\sigma_1,\sigma_2)\sigma'_3+E(\sigma_1,\sigma_2){\sigma'_3}^2=D(\sigma_1,\sigma_2)\sigma_3+E(\sigma_1,\sigma_2)\sigma_3^2,$$

where $C(s,t)$, $D(s,t)$ and $E(s,t)$ are as in Lemma 4.
We note that $E(\sigma_1,\sigma_2)=-3(-8+3\sigma_1)/360\neq 0$ for any integer σ_1.
Then the above equation has two solution with respect to σ'_3.
The first solution is $\sigma'_3=\sigma_3$ and the second one

$$\sigma'_3=-\frac{D(\sigma_1,\sigma_2)}{E(\sigma_1,\sigma_2)}-\sigma_3.$$

Proof of the Main Theorem

In the first case the theorem is proved. So we assume the second case. Since $B(\sigma_1, \sigma_2) = 0$ we have $\sigma_2 = (3 - 4\sigma_1 + \sigma_1^2)/3$. Hence

$$\sigma_3' = \frac{1}{729}\left(2(425 - 357\sigma_1 - 144\sigma_1^2 + 27\sigma_1^3) - \frac{490}{-8 + 3\sigma_1}\right) - \sigma_3. \qquad (*)$$

Proof of the Main Theorem

Since σ_3 and σ_3' are integers we obtain

(i) $N = 2(425 - 357\sigma_1 - 144\sigma_1^2 + 27\sigma_1^3) - \dfrac{490}{-8 + 3\sigma_1}$ is divisible by 729;

(ii) $-8 + 3\sigma_1$ is a divisor of 490;

(iii) $\sigma_2 = (3 - 4\sigma_1 + \sigma_1^2)/3$ is a positive integer.

There only finite number possibilities to satisfy these three conditions $\sigma_1 \in \{6, 19, 166\}$.
The case $\sigma_1 = 6$ can be excluded since we suggested that $\sigma_1 > 6$.

Proof of the Main Theorem

Two cases remained $\sigma_1 = 19$ and $\sigma_1 = 166$. Here by $(*)$ we have $\sigma_3' = 348 - \sigma_3$ and $\sigma_3' = 327789 - \sigma_3$ respectively. The respective values of σ_2 are 96 and 8965.
Let $\sigma_1 = 19$. We have

$$k + l + m = 19,\ \ k\,l + l\,m + m\,k = 96,\ \ k\,l\,m = \sigma_3.$$

This system has only one solution $\{k, l, m\} = \{3, 4, 12\}, \sigma_3 = 144$.
Now we are able to find parameters k', l', m', σ_3' of the graph $G' = \Theta(k', l', m')$. First of all, $\sigma_3' = 348 - \sigma_3 = 204$. Then we have

$$k' + l' + m' = 19, k'l' + l'm' + m'k = 96, k'l'm' = 204.$$

The latter system has no integer solutions. So the case $\sigma_1 = 19$ is impossible.

Let $\sigma_1 = 166$. We have

$$k + l + m = 166, \;\; k\,l + l\,m + m\,k = 8965, \;\; k\,l\,m = \sigma_3.$$

This system has only one solution $\{k, l, m\} = \{39, 59, 68\}, \sigma_3 = 39 \cdot 59 \cdot 68$. Find parameters k', l', m', σ_3' of the graph $G' = \Theta(k', l', m')$. Now, $\sigma_3' = 327789 - \sigma_3 = 171321$. Then we have

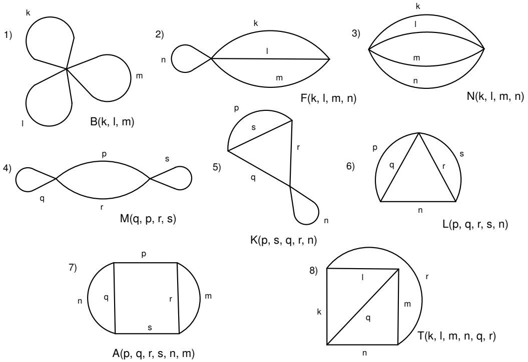
$$k' + l' + m' = 166, \;\; k'l' + l'm' + m'k' = 8965, \;\; k'l'm' = 171321.$$

The system has no integer solutions. The case $\sigma_1 = 166$ is also impossible. The proof of the theorem is finished.

Classification of genus three graphs

Theorem

Let G be a finite connected bridgeless genus three multigraph. Then G is isomorphic to the graph of one of eight types.



Let $V(G) = \{v_1, v_2, ..., v_i, ..., v_N\}$. Since there is no bridges then $d_i = deg(v_i) \geq 2 \ \forall i$. If we remove a vertex u, $deg(u) = 2$, the number of vertices and edges decreases by one respectively $\Rightarrow$ removal of the vertex which degree is equal two won't change the genus of graph. So we could suppose that $d_i \geq 3 \ \forall i$. $\sum_i^N d_i = 2E, \ E - V + 1 = g(G) = 3 \Rightarrow E = V + 2, \ V = 3N \leq \sum_i^N d_i = 2E = 2V + 4 \Rightarrow V \leq 4$

Isomorphism of isospectral genus three

Hypothesis

Two bridgeless genus three graphs belonging to the same type are isospectral if and only if they are isomorphic.

Graph B(k, l, m)

Two graphs $B(k, l, m)$ and $B(k', l', m')$ are isomorphic $\Leftrightarrow$ the unordered triples $\{k, l, m\}$ and $\{k', l', m'\}$ coincide. Equivalently,

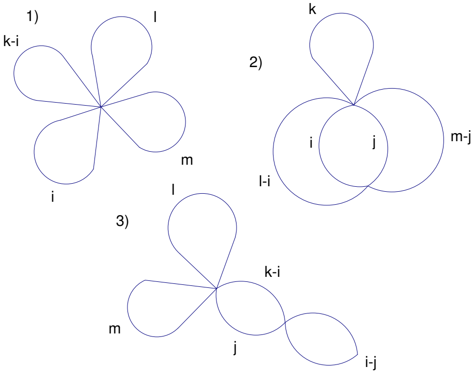
- $\sigma_1(k, l, m) = \sigma_1(k', l', m')$
- $\sigma_2(k, l, m) = \sigma_2(k', l', m')$
- $\sigma_3(k, l, m) = \sigma_3(k', l', m')$,

where $\sigma_1(x, y, z) = x + y + z$; $\sigma_2(x, y, z) = xy + yz + xz$; $\sigma_3(x, y, z) = xyz$

$B = B(k, l, m)$
$\mu(B, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^i c_i x^{n-i} + \ldots + (-1)^{n-1} c_{n-1} x$ - Laplacian polynomial of B. The number of vertices, edges and spanning trees of graph B could be calculated by

- $V(B) = k + l + m - 2$
- $E(B) = k + l + m$
- $T(B) = klm$

c_{n-2} for graph B



The number spanning trees for the graphs obtained from graph B by merging two vertices:

- $H1[k,l,m]=\sum_{i=1}^{k-1} iklm$
- $T1=H1[k,l,m]+H1[l,m,k]+H1[m,k,l]$

- $H2[k,l,m]=\sum_{i=1}^{l-1}\sum_{j=1}^{m-1} k(mi(l-i)+lj(m-j))$
- $T2=H2[k,l,m]+H2[l,m,k]+H2[m,k,l]$

- $H3[k,l,m]=\sum_{i=2}^{k-1}\sum_{j=1}^{i-1} ml(i-j)(k-i+j)$
- $T3=H3[k,l,m]+H3[l,m,k]+H3[m,k,l]$

If graphs B and B' are isospectral then

- $n = V(B) = V(B') = n' \Rightarrow k + l + m = k' + l' + m' \Rightarrow \sigma_1 = \sigma_1^{'}$
- $c_1(B) = c_1(B') \Rightarrow V(B)T(B) = V(B')T(B') \Rightarrow (\sigma_1 - 2)\,\sigma_3 = \left(\sigma_1^{'} - 2\right)\sigma_3^{'} \Rightarrow \sigma_3 = \sigma_3^{'}$

We can prove that $\sigma_2 = \sigma_2^{'}$ by calculation of c_{n-2}.

For graph B : $c_{n-2} = 1/12klm(12 - 5k - 4k^2 + k^3 - 5l + 2k^2 l - 4l^2 + 2kl^2 + l^3 - 5m + 2k^2 m + 2l^2 m - 4m^2 + 2km^2 + 2lm^2 + m^3)$ We can rewrite this using symmetric polynomials:

$$c_{n-2} = 1/12(12\sigma_3 - 5\sigma_1\sigma_3 - 4\sigma_1^2\sigma_3 + \sigma_1^3\sigma_3 + 8\sigma_2\sigma_3 - \sigma_1\sigma_2\sigma_3 - 3\sigma_3^2)$$

Previously, we proved that $c_{n-2}(B) = c_{n-2}(B')$. Taking into account that $\sigma_1 = \sigma_1^{'}$ and $\sigma_3 = \sigma_3^{'}$ we obtain $\sigma_2 = \sigma_2^{'}$. We proved that three symmetric polynomials coincide $\Longrightarrow$ the unordered triples (k, l, m) and (k', l', m') coincide $\Longrightarrow$ graphs B and B' are isomorphic.

Final remarks

1°. The Theorem is not valid for genus two graph with bridges.

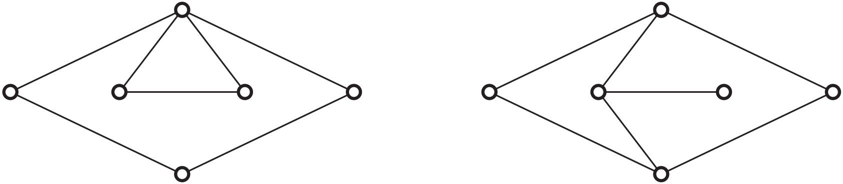


Fig. 4. Genus 2 graphs with the same spectrum. The second has a bridge.

The graphs share the following Laplacian polynomial:

$$-72x + 192x^2 - 176x^3 + 73x^4 - 14x^5 + x^6.$$

2°. There are isospectral bridgeless graphs of genus three which are not isomorphic.

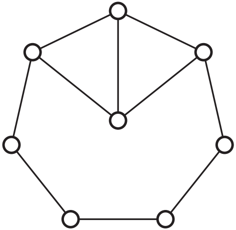


Fig. 5. Two nonisomorphic isospectral graphs of genus 3.

These two graph share the following Laplacian polynomial:

$$-384x + 1520x^2 - 2288x^3 + 1715x^4 - 708x^5 + 164x^6 - 20x^7 + x^8.$$

3°. Any bridgeless graph of genus one is isomorphic to a cyclic graph C_n for some $n \geq 1$. If two cyclic graphs C_m and C_n are isospectral then their Laplace polynomials are of the same degree $m = n$. Hence, the graphs are isomorphic.
In the same time, there are isospectral genus one graphs with bridges that are non-isomorphic.

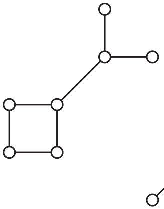


Fig. 6. Isospectral genus 1 graphs with bridges.

4°. One can hear genus of a graph. That is genus of a graph G is completely determined by its Laplace spectrum. Indeed, $g(G) = 1 - V(G) + E(G)$. Let

$$\mu(G, x) = x^n - c_1 x^{n-1} + \ldots + (-1)^{n-1} c_{n-1} x$$

be the Laplacian polynomial of G. By the arguments from the proof of Lemma 3.2 we have $n = V(G)$ and $c_1 = 2E(G)$. Thus $V(G)$ and $E(G)$, as well as the genus, are uniquely determined by the Laplacian polynomial.

5°. One cannot hear a bridge of a graph. Indeed, the two graphs on Fig. 4 are isospectral. We are not able to recognise the existence of a bridge of the second graph by its spectrum.