Nowhere-zero 3-flows in Cayley graphs of nilpotent groups

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Maps and Riemann Surfaces

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Nowhere-zero flows

A *nowhere-zero k-flow* on a graph G is an assignment of a *direction* and a *value* from $\{\pm 1, \pm 2, \ldots, \pm(k-1)\}$ to each edge of G
so that at each vertex

$$\text{flow in} \ = \ \text{flow out.}$$

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Question

What is the smallest value of m for which G has a nowhere-zero m-flow?

Group-valued flows

Let A be an abelian group. A *nowhere-zero A-flow* on a graph G is an assignment of a *direction* and a *value* from $A - 0$ to each edge of G so that at each vertex

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Group-valued flows

Let A be an abelian group. A *nowhere-zero A-flow* on a graph G is an assignment of a *direction* and a *value* from $A - 0$ to each edge of G so that at each vertex

$$\text{flow in} = \text{flow out}.$$

Theorem (Tutte, 1950)

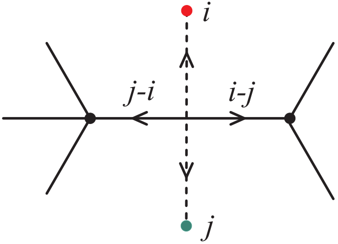
For every graph G the following statements are equivalent.

- *G has a nowhere-zero k-flow.*
- *G has a nowhere-zero $\mathbb{Z}_k$-flow.*
- *G has a nowhere-zero A-flow, where $|A| = k$.*

Flows and face-colourings

Theorem (Tutte, 1949)

Let K be a graph 2-cell embedded in an orientable surface S. If the embedding is m-face-colourable, then K admits a nowhere-zero m-flow. If S is the 2-sphere, the converse holds as well.



Tutte's flow conjectures

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- **4-Flow Conjecture** (1966): Every bridgeless graph with no Petersen minor has a nowhere-zero 4-flow.
- **3-Flow Conjecture** (1972): Every bridgeless graph with no 3-edge-cut has a nowhere-zero 3-flow.

Known results – 5FC

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Every bridgeless graph has a nowhere-zero 6-flow.

- 5-FC has been verified for various classes of graphs (but remains widely open)
- the conjecture reduces to verification on *snarks* ('non-trivial' cubic graphs that fail to have a 3-edge-colouring; equivalently, nowhere-zero 4-flow)
- the smallest counterexample must be a cyclically 6-connected snark of girth ≥ 9 (Kochol, 2006)

Known results – 4FC

- **Petersen Minor Conjecture**: Every bridgeless graph cubic graph with no Petersen minor is 3-edge-colourable.

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- **Cubic 4-Flow Conjecture**: Every bridgeless graph cubic graph with no Petersen minor has a nowhere-zero 4-flow.

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Known results – 4FC

- **Cubic 4-Flow Conjecture**: Every bridgeless graph cubic graph with no Petersen minor has a nowhere-zero 4-flow.
- C4FC is equivalent to its restriction to a class of almost planar graphs consisting of 2-connected apex graphs and double-cross graphs [Robertson, Seymour, Thomas, 1997]
- The authors announced that they had proved the restricted conjecture, thereby establishing the C4FC.

Known results – 3FC

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3FC has been verified e.g. for

- projective planar graphs (Steinberg & Younger, 1989)
- Cartesian products (Imrich & Škrekovski, 2003; Shu & Zhang, 2005)
- random graphs (Sudakov, 2001)

and reduced to 5-edge-connected 5-regular graphs (Zhang, Kochol, 2002).

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Theorem (Thomassen, 2012)

Every 8-edge-connected graph admits a nowhere-zero 3-flow.

Flows and symmetry in graphs

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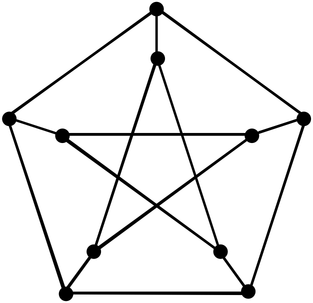

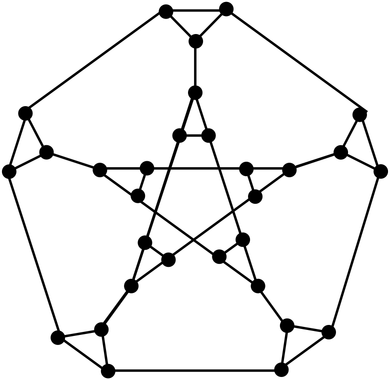

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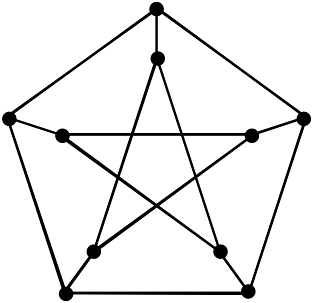

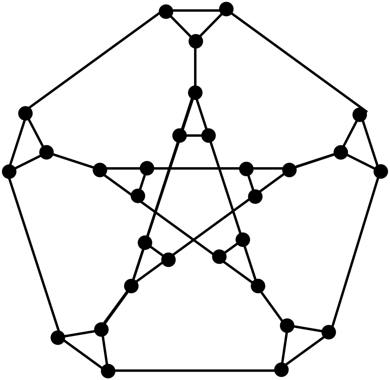

Natural question: What is the effect of graph symmetry on the existence of nowhere-zero flows on graphs?

Conjecture 1 (Lovász, 1969)

Every connected vertex-transitive graph has a hamilton path.

Flows and symmetry in graphs

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Conjecture 3 (Alspach et al., 1996)

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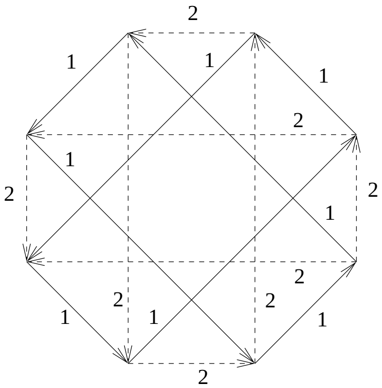
Conjecture 3 (Babai, 1995)

For some $c > 0$, there are infinitely many vertex-transitive graphs G, even Cayley graphs, without cycles of length $> (1 - c)|G|$.

Cayley graphs



Definition. *Cayley graph* $\mathrm{Cay}\,(G, S)$ of a group G with connection set S

- vertices . . . elements of G
- edges . . . $\{g, h\} \Leftrightarrow g^{-1}h \in S$ (where $S^{-1} = S$)

4-Flows on Cayley graphs

Theorem (Alspach et al., 1996)

Every Cayley graph (of valency ≥ 2) on a solvable group has a nowhere-zero 4-flow.

Easy for graphs of valency ≥ 4 (by Jaeger's 4-Flow Theorem)

$\Rightarrow$ crucial case: cubic graphs

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$\Rightarrow$ crucial case: cubic graphs

Theorem (Alspach et al., 1996)

Every cubic Cayley graph on a solvable group is 3-edge-colourable, and so has a nowhere-zero 4-flow.

Cayley snarks

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Two types of cubic Cayley graphs:

Type I: $\mathrm{Cay}\,(G, S)$ with $S = \{x, y, z;\ x^2 = y^2 = z^2 = 1\}$

Type II: $\mathrm{Cay}\,(G, S)$ with $S = \{r, l;\ r^n = l^2 = 1\}$

Cayley snarks

Theorem (Nedela and S., 2000)

If there exists a Cayley snark $\mathrm{Cay}(G, S)$*, then there is one such that either*

- G *is a simple non-abelian group, or*
- G *is "almost" simple non-abelian, i.e., it has exactly one proper normal subgroup* H*, where* $|G : H| = 2$*, and* H *is either simple non-abelian, or a direct product of two such groups.*

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Related result (Potočnik, 2004): *Every cubic graph admitting a vertex-transitive action of a solvable group – other than the Petersen graph – has a nowhere-zero 4-flow.*

3-Flows on Cayley graphs

The 3FC and the conjecture of Alspach et al. suggest

Conjecture

Every Cayley graph of valency ≥ 4 admits a nowhere-zero 3-flow.

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Conjecture

Every Cayley graph of valency ≥ 4 admits a nowhere-zero 3-flow.

It is sufficient to consider $\mathrm{Cay}\,(G, S)$ where G has even order and S contains an involution.

Theorem (Potočnik, Škrekovski & S., 2005)

Every Cayley graph of valency ≥ 4 on an abelian group admits a nowhere-zero 3-flow.

3-Flows on Cayley graphs

Theorem (Nánásiová and S.)

Let $K = \mathrm{Cay}\,(G, S)$ be a Cayley graph of valency ≥ 4 where $G = U \times H$ and U is a Sylow 2-subgroup of G. Then K has a nowhere-zero 3-flow.

Corollary

Every Cayley graph of valency ≥ 4 on a nilpotent group admits a nowhere-zero 3-flow.

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Definition. A *Sylow p-subgroup* of a group G is any maximal p-subgroup of G w.r.t. inclusion.

(Given a prime p dividing $|G|$, then a Sylow p-subgroup always exist, and any two Sylow p-subgroups are conjugate.)

I. Cayley graphs with central involutions

Theorem

If S contains a central involution of G, then $\mathrm{Cay}\,(G, S)$ has a nowhere-zero 3-flow.

Outline of proof: two steps

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If S contains a central involution of G, then $\mathrm{Cay}\,(G, S)$ has a nowhere-zero 3-flow.

II. Lifting of flows

- Cayley graph $\mathrm{Cay}\,(G, S)$
- normal subgroup $H \trianglelefteq G$ s.t. $H \cap S = \emptyset$

Lemma

If $\mathrm{Cay}\,(G/H, S/H)$ has a nowhere-zero 3-flow, then so has $\mathrm{Cay}\,(G, S)$.

Step I: Flows in Cayley graphs with central involutions

Let $\mathrm{Cay}\,(G, S)$ be a Cayley graph with $c \in S$ a central involution of G

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- Take $S_1 \subseteq S$ and $S_2 \subseteq S$ with $|S_1| = 3 = |S_2|$
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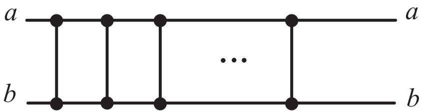
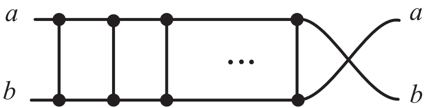
What can we say about these graphs?

Components of each $\mathrm{Cay}\,(G, S_i)$ are closed ladders



circular ladder



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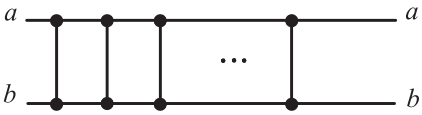

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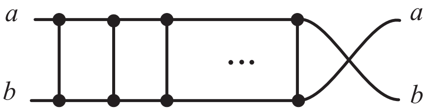

Moebius ladder

Lemma

Every closed ladder has a nowhere-zero 3-flow, or has a 3-flow with a single zero edge which can be chosen to be any spoke.

Step I: Flows in Cayley graphs with central involutions

- Each $\mathrm{Cay}\,(G, S_i)$ consists of isomorphic closed ladders
- Spokes of both $\mathrm{Cay}\,(G, S_1)$ and of $\mathrm{Cay}\,(G, S_2)$ are c-edges
- Every c-edge of $\mathrm{Cay}\,(G, S)$ belongs to exactly two closed ladders

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CONSTRUCTION OF FLOW

Construct a sequence

$$L_1,\ L_2,\ \ldots,\ L_r$$

of closed ladders along with a sequence

$$\phi_1,\ \phi_2,\ \ldots,\ \phi_r$$

of 3-flows with at most one zero edge which fill up the whole $\mathrm{Cay}\,(G, S)$.

Step II: Induction in the Cayley group

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Final proof

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Proof.

Let U be a Sylow 2-subgroup of G. Employ induction on $|U|$.

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Let U be a Sylow 2-subgroup of G. Employ induction on $|U|$.

- Induction basis: $|U| \leq 2$

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- Induction step: Let $|U| > 2$.
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- Induction step: Let $|U| > 2$.
  Then U contains a central involution c of G.
- If $c \in S$, use the previous result.
- If $c \notin S$, take $\mathrm{Cay}\,(G/\langle c\rangle, S/\langle c\rangle)$. Clearly, $U/\langle c\rangle$ is a Sylow 2-subgroup of $G/\langle c\rangle$, and $|U/\langle c\rangle| < |U|$. Now apply the hypothesis.

□

THANK YOU!