Algebraic Theory of SKEW-MORPHISMS

Robert Jajcay
Comenius University
robert.jajcay@fmph.uniba.sk

....

Sobolev Institute 2014

November 4, 2014

(Cayley) Regular Action

Cayley: A finite group G is isomorphic to a subgroup of $\mathbb{S}_G$:

$$\sigma : G \to \mathbb{S}_G; \qquad \sigma_g(h) = gh, \quad \text{for all } h \in G$$

(Cayley) Regular Action

Cayley: A finite group G is isomorphic to a subgroup of $\mathbb{S}_G$:

$$\sigma : G \to \mathbb{S}_G; \qquad \sigma_g(h) = gh, \quad \text{for all } h \in G$$

$$G \cong G_L$$

left regular representation

Cayley Graphs

Given a left-regular representation of a finite group G_L,

Cayley Graphs

Given a left-regular representation of a finite group G_L, consider the induced action on unordered pairs $\binom{G}{2}$, and let $\mathcal{E}$ be a union of orbits of G_L on $\binom{G}{2}$.

Cayley Graphs

Given a left-regular representation of a finite group G_L, consider the induced action on unordered pairs $\binom{G}{2}$, and let $\mathcal{E}$ be a union of orbits of G_L on $\binom{G}{2}$.

Definition

The graph $\Gamma = (G, \mathcal{E})$ is called a **Cayley graph**, and

$$G_L \leq Aut(\Gamma)$$

Given a left-regular representation of a finite group G_L, consider the induced action on unordered pairs $\binom{G}{2}$, and let $\mathcal{E}$ be a union of orbits of G_L on $\binom{G}{2}$.

Definition

The graph $\Gamma = (G, \mathcal{E})$ is called a **Cayley graph**, and

$$G_L \leq Aut(\Gamma)$$

$Aut(\Gamma)$ may be bigger than G_L:

$$Aut(\Gamma) = G_L \cdot Stab_{Aut(\Gamma)}(u)$$

Cayley Graphs

Equivalent definition: Let G be a finite group and X be a subset of G that does not contain the identity 1_G and is closed under taking inverses:

The **Cayley graph** $\Gamma = C(G, X)$ is the graph with vertex set $V = G$ and edge set $\mathcal{E} = \{ \ \{g, gx\} \mid g \in G, x \in X \ \}$.

Cayley Graphs

Equivalent definition: Let G be a finite group and X be a subset of G that does not contain the identity 1_G and is closed under taking inverses:

The **Cayley graph** $\Gamma = C(G, X)$ is the graph with vertex set $V = G$ and edge set $\mathcal{E} = \{ \ \{g, gx\} \ | \ g \in G, x \in X \ \}$.

$Stab_{Aut(\Gamma)}(1_G)$ preserves X.

- if G is abelian and X contains at least one non-involution, then $\varphi : x \to x^{-1}$ is a (non-trivial) group automorphism of G that preserves X (since X is closed under inverses):

$$\varphi \in Stab_{Aut(\Gamma)}(1_G)$$

- if G is abelian and X contains at least one non-involution, then $\varphi : x \to x^{-1}$ is a (non-trivial) group automorphism of G that preserves X (since X is closed under inverses):

$$\varphi \in Stab_{Aut(\Gamma)}(1_G)$$

- Cayley graphs $C(G, X)$ for which $Stab_{Aut(\Gamma)}(1_G)$ is non-trivial and G_L is normal in $Aut(\Gamma)$ are called **normal Cayley graphs**

Graphical Regular Representation

Definition

A Cayley graph $\Gamma = (G, X)$ is a **graphical regular representation** of the group G iff $Aut(\Gamma) = G_L$

Graphical Regular Representation

Definition

A Cayley graph $\Gamma = (G, X)$ is a **graphical regular representation** of the group G iff $Aut(\Gamma) = G_L$ iff $Stab_{Aut(\Gamma)}(1_G) = \langle id \rangle$.

Graphical Regular Representation

Definition

A Cayley graph $\Gamma = (G, X)$ is a **graphical regular representation** of the group G iff $Aut(\Gamma) = G_L$ iff $Stab_{Aut(\Gamma)}(1_G) = \langle id \rangle$.

Abelian groups of exponent greater than 2 do not admit GRR's.

Graphical Regular Representation

Theorem (Godsil, Watkins, ...)

Let G be a finite group that does not have a GRR, i.e., a finite group that does not admit a regular representation as the full automorphism group of a graph. Then G is an abelian group of exponent greater than 2 or G is a generalized dicyclic group $\langle a, x \mid a^{2n} = 1, x^2 = a^n, x^{-1}ax = a^{-1} \rangle$ or G is isomorphic to one of the 13 groups : $\mathcal{Z}_2^2$, $\mathcal{Z}_2^3$, $\mathcal{Z}_2^4$, $\mathcal{D}_3$, $\mathcal{D}_4$, $\mathcal{D}_5$, $\mathcal{A}_4$, $\mathcal{Q} \times \mathcal{Z}_3$, $\mathcal{Q} \times \mathcal{Z}_4$,
$\langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ abc = bca = cab \rangle$,
$\langle a, b \mid a^8 = b^2 = 1,\ b^{-1}ab = a^5 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^2 = 1,\ ab = ba,\ (ac)^2 = (bc)^2 = 1 \rangle$,
$\langle a, b, c \mid a^3 = b^3 = c^3 = 1,\ ac = ca,\ bc = cb,\ b^{-1}ab = ac \rangle$.

Proportion of GRR's among Cayley graphs

Conjecture:

Almost all Cayley graphs are GRR's.

Proportion of GRR's among Cayley graphs

Conjecture:

Almost all Cayley graphs are GRR's.

$$\lim_{n\to\infty} \frac{\#\text{ of GRR's of order } \leq n}{\#\text{ of Cayley graphs of order } \leq n} = 1$$

Proportion of GRR's among Cayley graphs

Conjecture:

Almost all Cayley graphs are GRR's.

$$\lim_{n\to\infty} \frac{\#\text{ of GRR's of order } \leq n}{\#\text{ of Cayley graphs of order } \leq n} = 1$$

True for nilpotent groups.

Orientable Maps

- an **orientable map** $\mathcal{M}$ is a 2-cell embedding of a graph in an orientable surface; an embedding in which every face is homeomorphic to the open disc

Orientable Maps

- an **orientable map** $\mathcal{M}$ is a 2-cell embedding of a graph in an orientable surface; an embedding in which every face is homeomorphic to the open disc
- given a finite connected graph $\Gamma = (V, E)$, an orientable embedding of Γ is determined by choosing a *cyclic local permutation* ρ_v of arcs emanating from each vertex:

$$\rho \in \mathbb{S}_{D(\Gamma)} : \quad \rho = \cup \rho_v, \; v \in V$$

Orientable Maps

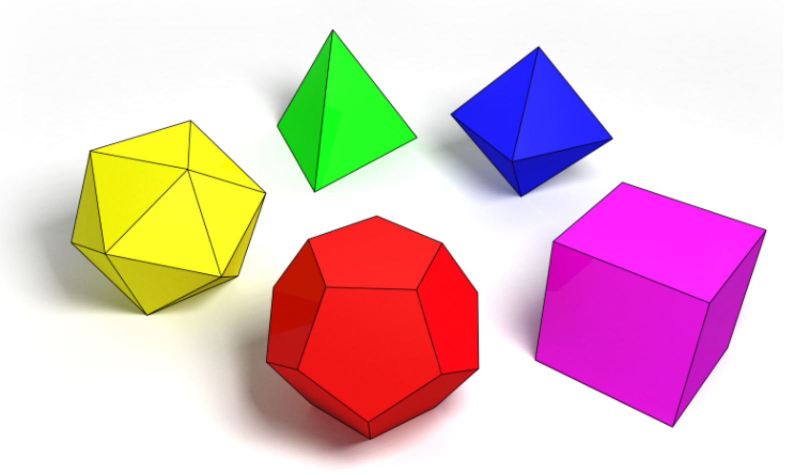
- an **orientable map** $\mathcal{M}$ is a 2-cell embedding of a graph in an orientable surface; an embedding in which every face is homeomorphic to the open disc
- given a finite connected graph $\Gamma = (V, E)$, an orientable embedding of Γ is determined by choosing a *cyclic local permutation* ρ_v of arcs emanating from each vertex:

$$\rho \in \mathbb{S}_{D(\Gamma)}: \quad \rho = \cup \rho_v, \ v \in V$$

- if $\Gamma = C(G, X)$ is a Cayley graph, ρ_g can be thought of as a cyclic permutation of X:

$$\rho_g((g, x)) = (g, \rho_g(x))$$

Classical Examples - The Five Platonic Solids



Four of the five platonic solids are embeddings of Cayley graphs

Definition

A **map automorphism** of an orientable map $\mathcal{M}$ is a permutation of the darts of the map that preserves adjacency and faces:

$$\varphi\lambda = \lambda\varphi, \quad \varphi\rho = \rho\varphi$$

Regular Maps

Definition

A **map automorphism** of an orientable map $\mathcal{M}$ is a permutation of the darts of the map that preserves adjacency and faces:

$$\varphi\lambda = \lambda\varphi, \quad \varphi\rho = \rho\varphi$$

The connectivity of the underlying graph of a map $\mathcal{M}$ implies that for any pair of darts e and f, there exists *at most* one automorphism of $\mathcal{M}$ mapping e to f.

Regular Maps

Definition

A **map automorphism** of an orientable map $\mathcal{M}$ is a permutation of the darts of the map that preserves adjacency and faces:

$$\varphi\lambda = \lambda\varphi, \quad \varphi\rho = \rho\varphi$$

The connectivity of the underlying graph of a map $\mathcal{M}$ implies that for any pair of darts e and f, there exists *at most* one automorphism of $\mathcal{M}$ mapping e to f.

Definition

An orientable map $\mathcal{M}$ is called **regular** if any pair of arcs admits the existence of an orientation preserving automorphism of $\mathcal{M}$ that maps the first arc to the second.

Regular Maps

Definition

A **map automorphism** of an orientable map $\mathcal{M}$ is a permutation of the darts of the map that preserves adjacency and faces:

$$\varphi\lambda = \lambda\varphi, \quad \varphi\rho = \rho\varphi$$

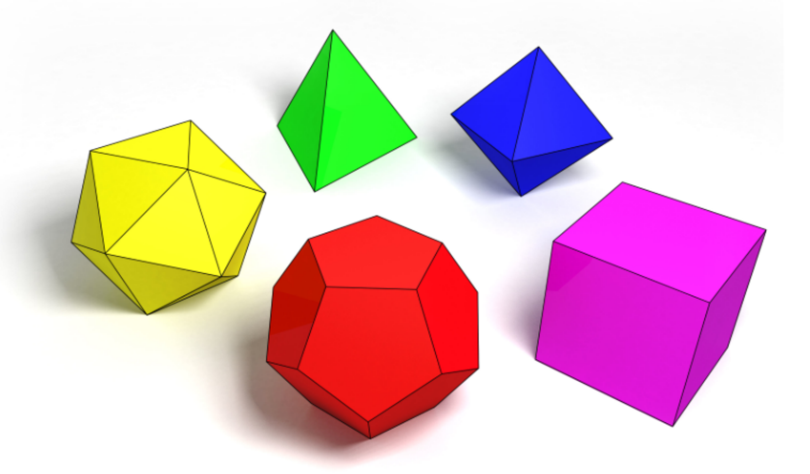
The connectivity of the underlying graph of a map $\mathcal{M}$ implies that for any pair of darts e and f, there exists *at most* one automorphism of $\mathcal{M}$ mapping e to f.

Definition

An orientable map $\mathcal{M}$ is called **regular** if any pair of arcs admits the existence of an orientation preserving automorphism of $\mathcal{M}$ that maps the first arc to the second.

A map $\mathcal{M}$ is regular if and only if $|Aut\mathcal{M}| = |D(\mathcal{M})|$

Classical Examples - The Five Platonic Solids



All five platonic solids are regular maps

Cayley Maps

Definition

A *Cayley map* $CM(G, X, p)$ is an embedding of a connected *Cayley graph*, $C(G, X)$, that has the same local orientation p at each vertex.

Cayley Maps

Definition

A *Cayley map* $CM(G, X, p)$ is an embedding of a connected *Cayley graph*, $C(G, X)$, that has the same local orientation p at each vertex.

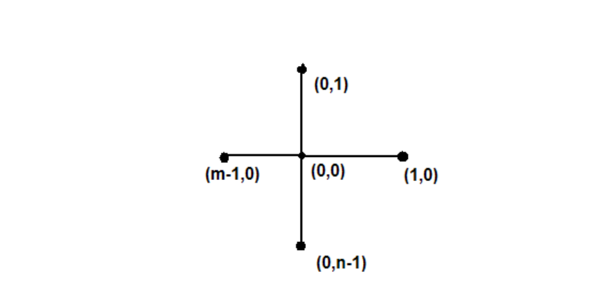
Equivalently, a Cayley map is a drawing of a Cayley graph on a surface such that the outgoing darts are ordered the same way around each vertex; the local successor of the dart (g, x) is the dart $(g, p(x))$.

Cayley Maps

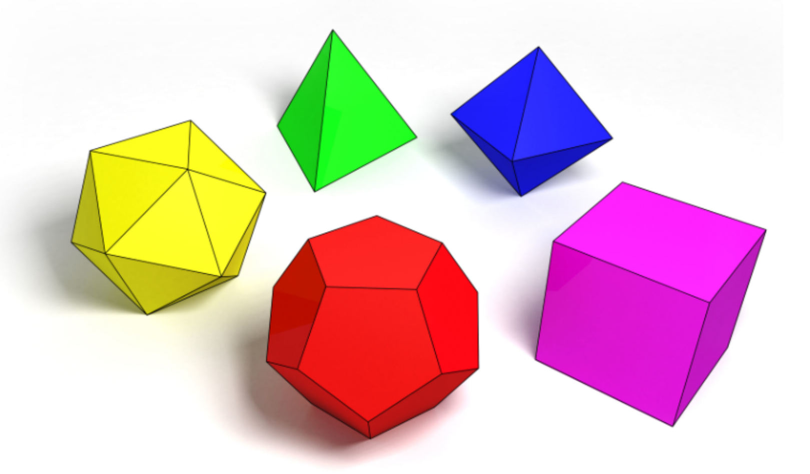
Definition

A *Cayley map* $CM(G, X, p)$ is an embedding of a connected *Cayley graph*, $C(G, X)$, that has the same local orientation p at each vertex.

Equivalently, a Cayley map is a drawing of a Cayley graph on a surface such that the outgoing darts are ordered the same way around each vertex; the local successor of the dart (g, x) is the dart $(g, p(x))$.



Classical Examples - The Five Platonic Solids



Four of the five platonic solids are Cayley maps

- all the left multiplication automorphisms of the underlying group induce map automorphisms of the Cayley map;

$$G_L \leq Aut(CM(G, X, p))$$

- all the left multiplication automorphisms of the underlying group induce map automorphisms of the Cayley map;

$$G_L \leq Aut(CM(G, X, p))$$

- $|\mathcal{D}(CM(G, X, p))| = |G| \cdot |X|$

- all the left multiplication automorphisms of the underlying group induce map automorphisms of the Cayley map;

$$G_L \leq Aut(CM(G,X,p))$$

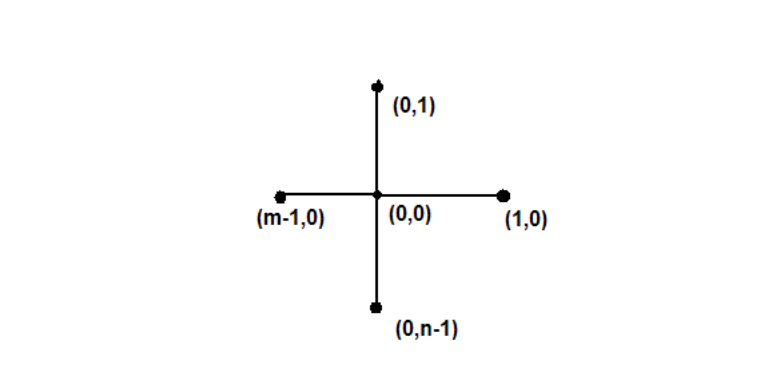
- $|\mathcal{D}(CM(G,X,p))| = |G| \cdot |X|$
- in order for a Cayley map to be regular, the stabilizer of any vertex in $Aut(CM(G,X,p))$ must be of size $|X|$

- all the left multiplication automorphisms of the underlying group induce map automorphisms of the Cayley map;

$$G_L \leq Aut(CM(G, X, p))$$

- $|\mathcal{D}(CM(G, X, p))| = |G| \cdot |X|$
- in order for a Cayley map to be regular, the stabilizer of any vertex in $Aut(CM(G, X, p))$ must be of size $|X|$
- since the stabilizers of orientable maps are cyclic, in order for a Cayley map to be regular, there must exist an automorphism Φ that maps $(1, x)$ to $(1, p(x))$

$$\Phi(1_G) = 1_G \text{ and } \Phi((1_G, x)) = (1_G, p(x))$$



Skew-Morphisms

Definition (RJ,Širáň)

A *skew-morphism* of a group G is a permutation φ of G preserving the identity and satisfying the property

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$$

for all $g, h \in G$ and a function $\pi : G \to \mathbb{Z}_{|\varphi|}$, called the *power function* of G.

Skew-Morphisms

Definition (RJ,Širáň)

A *skew-morphism* of a group G is a permutation φ of G preserving the identity and satisfying the property

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$$

for all $g, h \in G$ and a function $\pi : G \to \mathbb{Z}_{|\varphi|}$, called the *power function* of G.

- skew-morphisms were originally introduced for the study of regular Cayley maps

Skew-Morphisms

Definition (RJ,Širáň)

A *skew-morphism* of a group G is a permutation φ of G preserving the identity and satisfying the property

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$$

for all $g, h \in G$ and a function $\pi : G \to \mathbb{Z}_{|\varphi|}$, called the *power function* of G.

- skew-morphisms were originally introduced for the study of regular Cayley maps
- they have since proved central in the theory of cyclic group extensions

Skew-Morphisms

Definition (RJ,Širáň)

A *skew-morphism* of a group G is a permutation φ of G preserving the identity and satisfying the property

$$\varphi(gh) = \varphi(g)\varphi^{\pi(g)}(h)$$

for all $g, h \in G$ and a function $\pi : G \to \mathbb{Z}_{|\varphi|}$, called the *power function* of G.

- skew-morphisms were originally introduced for the study of regular Cayley maps
- they have since proved central in the theory of cyclic group extensions
- the focus of this talk is on the interplay between their original use in the topological graph theory and their group-theoretical properties

Skew-Morphisms

Theorem (RJ,Širáň)

Let $\mathcal{M} = CM(G, X, p)$ be any Cayley map. Then $\mathcal{M}$ is regular iff there exists a skew-morphism φ of G satisfying the property $\varphi(x) = p(x)$ for all $x \in X$.

Skew-Morphisms

Theorem (RJ,Širáň)

Let $\mathcal{M} = CM(G, X, p)$ be any Cayley map. Then $\mathcal{M}$ is regular iff there exists a skew-morphism φ of G satisfying the property $\varphi(x) = p(x)$ for all $x \in X$.

Proof.

- If $\Phi(1_G) = 1_G$ and $\Phi((1_G, x)) = (1_G, p(x))$ is the map automorphism generating the stabilizer of 1_G, then the mapping $\varphi : G \to G$ induced by Φ on G is a skew-morphism satisfying the properties $\varphi(1_G) = 1_G$ and $\varphi(x) = p(x)$.

Skew-Morphisms

Theorem (RJ,Širáň)

Let $\mathcal{M} = CM(G, X, p)$ be any Cayley map. Then $\mathcal{M}$ is regular iff there exists a skew-morphism φ of G satisfying the property $\varphi(x) = p(x)$ for all $x \in X$.

Proof.

- If $\Phi(1_G) = 1_G$ and $\Phi((1_G, x)) = (1_G, p(x))$ is the map automorphism generating the stabilizer of 1_G, then the mapping $\varphi : G \to G$ induced by Φ on G is a skew-morphism satisfying the properties $\varphi(1_G) = 1_G$ and $\varphi(x) = p(x)$.
- If $\varphi : G \to G$ satisfying the properties $\varphi(1_G) = 1_G$ and $\varphi(x) = p(x)$ is a skew-morphism of G, then Φ defined by $\Phi(g, x) = (\varphi(g), \varphi(g)^{-1}\varphi(gx))$ is the required map automorphism.

Constructions of regular Cayley maps

In order to construct all regular Cayley maps for a given group G:

- every regular Cayley map on G is of the form

$$CM(G, \{x, \varphi(x), \ldots, \varphi^{n-1}(x)\}, (x, \varphi(x), \ldots, \varphi^{n-1}(x))),$$

  where φ is a skew-morphisms with a **generating orbit** $\{x, \varphi(x), \ldots, \varphi^{n-1}(x)\}$ **that is closed under inverses**

Constructions of regular Cayley maps

In order to construct all regular Cayley maps for a given group G:

- every regular Cayley map on G is of the form

$$CM(G, \{x, \varphi(x), \ldots, \varphi^{n-1}(x)\}, (x, \varphi(x), \ldots, \varphi^{n-1}(x))),$$

  where φ is a skew-morphisms with a **generating orbit** $\{x, \varphi(x), \ldots, \varphi^{n-1}(x)\}$ **that is closed under inverses**
- the order of φ, the order of Φ, and the order of the stabilizer of 1_G in the automorphism group of a regular Cayley map, are all equal

Constructions of regular Cayley maps

In order to construct all regular Cayley maps for a given group G:

- every regular Cayley map on G is of the form

$$CM(G, \{x, \varphi(x), \dots, \varphi^{n-1}(x)\}, (x, \varphi(x), \dots, \varphi^{n-1}(x))),$$

  where φ is a skew-morphisms with a **generating orbit** $\{x, \varphi(x), \dots, \varphi^{n-1}(x)\}$ **that is closed under inverses**
- the order of φ, the order of Φ, and the order of the stabilizer of 1_G in the automorphism group of a regular Cayley map, are all equal
- the order of φ in this case equals the length of its generating orbit that is closed under inverses

Lemma (RJ, Širáň)

Let φ *be a skew-morphism of a group* G *and let* π *be the power function of* φ. *Then the following holds :*

1. *the set* $Ker\varphi = \{g \in G \mid \pi(g) = 1\}$ *is a subgroup of* G*;*
2. $\pi(g) = \pi(h)$ *if and only if* g *and* h *belong to the same right coset of the subgroup* $Ker\varphi$ *in* G.

Algebraic Properties of Skew-Morphisms

Lemma (RJ, Širáň)

Let φ be a skew-morphism of a group G and let π be the power function of φ. Then the following holds :

1. *the set* $Ker\varphi = \{g \in G \mid \pi(g) = 1\}$ *is a subgroup of* G*;*
2. $\pi(g) = \pi(h)$ *if and only if* g *and* h *belong to the same right coset of the subgroup* $Ker\varphi$ *in* G*.*

Note: The kernel of φ that gives rise to a regular Cayley map is always non-trivial.

Lemma (RJ, Širáň)

Let φ be a skew-morphism of a group G and let π be the power function of φ. Then the following holds :

1. *the set $Ker\varphi = \{g \in G \mid \pi(g) = 1\}$ is a subgroup of G;*
2. *$\pi(g) = \pi(h)$ if and only if g and h belong to the same right coset of the subgroup $Ker\varphi$ in G.*

Note: The kernel of φ that gives rise to a regular Cayley map is always non-trivial.

Lemma (Conder, RJ, Tucker)

If A is a finite abelian group and φ is a skew-morphism of A, then

1. *φ preserves $Ker\,\pi$ setwise;*
2. *the restriction of φ to $Ker\,\pi$ is a group automorphism.*

The interplay between the local permutation and the structure of the group

Definition

Let $\mathcal{M} = CM(G, X, p)$ be a Cayley map. The function $\chi : X \to \mathbb{Z}_{|X|}$ defined by the rule that $\chi(x)$ is the smallest non-negative integer with the property $p^{\chi(x)}(x) = x^{-1}$, is called the **distribution of inverses of the map** $\mathcal{M}$.

The interplay between the local permutation and the structure of the group

Definition

Let $\mathcal{M} = CM(G, X, p)$ be a Cayley map. The function $\chi : X \to \mathbb{Z}_{|X|}$ defined by the rule that $\chi(x)$ is the smallest non-negative integer with the property $p^{\chi(x)}(x) = x^{-1}$, is called the **distribution of inverses of the map** $\mathcal{M}$.

Definition

- If the cyclic permutation p of a Cayley map $\mathcal{M} = CM(G, X, p)$ satisfies the identity $p(x^{-1}) = (p(x))^{-1}$, for all $x \in X$, we say that the map is a **balanced Cayley map**.

The interplay between the local permutation and the structure of the group

Definition

Let $\mathcal{M} = CM(G, X, p)$ be a Cayley map. The function $\chi : X \to \mathbb{Z}_{|X|}$ defined by the rule that $\chi(x)$ is the smallest non-negative integer with the property $p^{\chi(x)}(x) = x^{-1}$, is called the **distribution of inverses of the map** $\mathcal{M}$.

Definition

- If the cyclic permutation p of a Cayley map $\mathcal{M} = CM(G, X, p)$ satisfies the identity $p(x^{-1}) = (p(x))^{-1}$, for all $x \in X$, we say that the map is a **balanced Cayley map**.
- If the cyclic permutation p of a Cayley map $\mathcal{M} = CM(G, X, p)$ satisfies the identity $p(x^{-1}) = (p^t(x))^{-1}$, for all $x \in X$, we say that the map is a ***t*-balanced Cayley map**.

Special Classes of Skew-Morphisms

- **group automorphisms**; $Ker\varphi = G$, $\varphi(ab) = \varphi(a)\varphi^1(b)$

Special Classes of Skew-Morphisms

- **group automorphisms**; $Ker\varphi = G$, $\varphi(ab) = \varphi(a)\varphi^{1}(b)$
- **antiautomorphisms**; $[G : Ker\varphi] = 2$, $\varphi(ab) = \varphi(a)\varphi^{\pm 1}(b)$

Special Classes of Skew-Morphisms

- **group automorphisms**; $Ker\varphi = G$, $\varphi(ab) = \varphi(a)\varphi^1(b)$
- **antiautomorphisms**; $[G : Ker\varphi] = 2$, $\varphi(ab) = \varphi(a)\varphi^{\pm 1}(b)$
- ***t*-balanced skew-morphisms**; $[G : Ker\varphi] = 2$, $\pi(a) \in \{1, t\}$

Special Classes of Skew-Morphisms

- **group automorphisms**; $Ker\varphi = G$, $\varphi(ab) = \varphi(a)\varphi^1(b)$
- **antiautomorphisms**; $[G : Ker\varphi] = 2$, $\varphi(ab) = \varphi(a)\varphi^{\pm 1}(b)$
- ***t*-balanced skew-morphisms**; $[G : Ker\varphi] = 2$, $\pi(a) \in \{1, t\}$
- if the power function of a skew-morphism assumes exactly i values in $\mathbb{Z}_{|\varphi|}$, then it is called **of skew-type *i***; $[G : Ker\varphi] = i$

Special Classes of Skew-Morphisms

- **group automorphisms**; $Ker\varphi = G$, $\varphi(ab) = \varphi(a)\varphi^1(b)$
- **antiautomorphisms**; $[G : Ker\varphi] = 2$, $\varphi(ab) = \varphi(a)\varphi^{\pm 1}(b)$
- ***t*-balanced skew-morphisms**; $[G : Ker\varphi] = 2$, $\pi(a) \in \{1, t\}$
- if the power function of a skew-morphism assumes exactly i values in $\mathbb{Z}_{|\varphi|}$, then it is called **of skew-type *i***; $[G : Ker\varphi] = i$
- **skew-morphisms whose orbits are contained within the cosets** of the kernel *(c.o.p.f.)*

Known Classifications

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)

Known Classifications

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)
- balanced skew-morphisms on cyclic, dihedral, and generalized quaternion groups that give rise to a regular Cayley map (Yan Wang and Rongquan Feng)

Known Classifications

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)
- balanced skew-morphisms on cyclic, dihedral, and generalized quaternion groups that give rise to a regular Cayley map (Yan Wang and Rongquan Feng)
- -1-balanced skew-morphisms on abelian groups that give rise to a regular Cayley map (M. Conder, RJ, T. Tucker)

Known Classifications

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)
- balanced skew-morphisms on cyclic, dihedral, and generalized quaternion groups that give rise to a regular Cayley map (Yan Wang and Rongquan Feng)
- -1-balanced skew-morphisms on abelian groups that give rise to a regular Cayley map (M. Conder, RJ, T. Tucker)
- t-balanced skew-morphisms of cyclic groups that give rise to a regular Cayley map (Young Soo Kwon)

Known Classifications

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)
- balanced skew-morphisms on cyclic, dihedral, and generalized quaternion groups that give rise to a regular Cayley map (Yan Wang and Rongquan Feng)
- -1-balanced skew-morphisms on abelian groups that give rise to a regular Cayley map (M. Conder, RJ, T. Tucker)
- t-balanced skew-morphisms of cyclic groups that give rise to a regular Cayley map (Young Soo Kwon)
- t-balanced skew-morphisms on dihedral groups that give rise to a regular Cayley map (Jin Ho Kwak, Young Soo Kwon, and Rongquan Feng)

Known Classifications

- all skew-morphisms of $\mathbb{Z}_p$ that give rise to a regular Cayley map are group automorphisms (RJ, Širáň)
- balanced skew-morphisms on cyclic, dihedral, and generalized quaternion groups that give rise to a regular Cayley map (Yan Wang and Rongquan Feng)
- -1-balanced skew-morphisms on abelian groups that give rise to a regular Cayley map (M. Conder, RJ, T. Tucker)
- t-balanced skew-morphisms of cyclic groups that give rise to a regular Cayley map (Young Soo Kwon)
- t-balanced skew-morphisms on dihedral groups that give rise to a regular Cayley map (Jin Ho Kwak, Young Soo Kwon, and Rongquan Feng)
- t-balanced skew-morphisms on dicyclic groups that give rise to a regular Cayley map (Jin Ho Kwak and Ju-Mok Oh)

Known Classifications

- t-balanced skew-morphisms on semi-dihedral groups that give rise to a regular Cayley map (Ju-Mok Oh)

Known Classifications

- t-balanced skew-morphisms on semi-dihedral groups that give rise to a regular Cayley map (Ju-Mok Oh)
- regular, non-balanced Cayley maps over a dihedral group D_{2n}, n odd (Kovács, Marušič, Muzychuk)

Known Classifications

- t-balanced skew-morphisms on semi-dihedral groups that give rise to a regular Cayley map (Ju-Mok Oh)
- regular, non-balanced Cayley maps over a dihedral group D_{2n}, n odd (Kovács, Marušič, Muzychuk)
- recent work of Jun-Yang Zhang suggests possibilities for classifying regular Cayley maps on cyclic groups with kernel of index 3

Skew-Morphisms from Cyclic Extensions

Let $G = A\langle\rho\rangle$, $A \cap \langle\rho\rangle = \langle 1_G\rangle$.

Skew-Morphisms from Cyclic Extensions

Let $G = A\langle\rho\rangle$, $A \cap \langle\rho\rangle = \langle 1_G \rangle$.

For every $a \in G$,

$$\rho a = a'\rho^i,$$

for some unique $a' \in A$ and some unique nonnegative integer i less than the order of ρ.

Skew-Morphisms from Cyclic Extensions

Let $G = A\langle\rho\rangle$, $A \cap \langle\rho\rangle = \langle 1_G \rangle$.

For every $a \in G$,

$$\rho a = a'\rho^i,$$

for some unique $a' \in A$ and some unique nonnegative integer i less than the order of ρ.

Define $\varphi(a) = a'$ and $\pi(a) = i$. Then for any a, b in A,

$$\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b).$$

Skew-Morphisms from Cyclic Extensions

Let $G = A\langle\rho\rangle$, $A \cap \langle\rho\rangle = \langle 1_G \rangle$.

For every $a \in G$,

$$\rho a = a' \rho^i,$$

for some unique $a' \in A$ and some unique nonnegative integer i less than the order of ρ.

Define $\varphi(a) = a'$ and $\pi(a) = i$. Then for any a, b in A,

$$\varphi(ab) = \varphi(a)\varphi^{\pi(a)}(b).$$

Already observed in the 1930's (e.g., Oystein Ore, 1938).

Cyclic Extensions from Skew-Morphisms

Let H be a group, and φ be a skew-morphism of H with power function π,

$$s(i, b) = \sum_{j=0}^{i-1} \pi(\varphi^j(b)).$$

Cyclic Extensions from Skew-Morphisms

Let H be a group, and φ be a skew-morphism of H with power function π,

$$s(i,b) = \sum_{j=0}^{i-1} \pi(\varphi^j(b)).$$

Define a multiplication $*$ on $H \times \langle \varphi \rangle$ as follows:

$$(a,\varphi^i) * (b,\varphi^j) = (a\varphi^i(b), \varphi^{s(i,b)+j}),$$

for all $a, b \in H$ and all $i, j \in \mathbb{Z}_{|\varphi|}$.

Cyclic Extensions from Skew-Morphisms

Let H be a group, and φ be a skew-morphism of H with power function π,

$$s(i,b) = \sum_{j=0}^{i-1} \pi(\varphi^j(b)).$$

Define a multiplication $*$ on $H \times \langle \varphi \rangle$ as follows:

$$(a, \varphi^i) * (b, \varphi^j) = (a\varphi^i(b), \varphi^{s(i,b)+j}),$$

for all $a, b \in H$ and all $i, j \in \mathbb{Z}_{|\varphi|}$.

Theorem (Conder,RJ,Tucker; Kovács and Nedela)

Let H be a group and φ be a skew-morphism of H of finite order m and power function π. Then $A = (H \times \langle \varphi \rangle, *)$ is a group, $H \times \langle \varphi \rangle$ is a complementary factorization of A, and the skew-morphism of H associated with this factorization is equal to φ.

Cyclic Extensions from Skew-Morphisms

Theorem (Conder, RJ, Tucker)

If G is any finite group with a complementary subgroup factorisation $G = AY$ with Y cyclic, then for any generator y of Y, the order of the skew morphism φ of A is the index in Y of its core in G, or equivalently, the smallest index in Y of a normal subgroup of G.
Moreover, in this case the quotient $\overline{G} = G/\mathrm{Core}_G(Y)$ is the skew product group associated with the skew morphism φ, with complementary subgroup factorisation $\overline{G} = \overline{A}\,\overline{Y}$ where $\overline{A} = AY/Y \cong A/(A \cap Y) \cong A$ and $\overline{Y} = Y/\mathrm{Core}_G(Y)$.

Cyclic Extensions from Skew-Morphisms

The **core of a skew-morphism** φ of a group A is the union of the orbits of φ that lie entirely in the kernel of φ.

Cyclic Extensions from Skew-Morphisms

The **core of a skew-morphism** φ of a group A is the union of the orbits of φ that lie entirely in the kernel of φ.

Theorem (Jun-Yang Zhang)

The core of a skew-morphism φ of A is a normal subgroup of $A \times \langle \varphi \rangle$ and it is the maximal normal subgroup of $A \times \langle \varphi \rangle$ contained in A.

Cyclic Extensions from Skew-Morphisms

The **core of a skew-morphism** φ of a group A is the union of the orbits of φ that lie entirely in the kernel of φ.

Theorem (Jun-Yang Zhang)

The core of a skew-morphism φ of A is a normal subgroup of $A \times \langle\varphi\rangle$ and it is the maximal normal subgroup of $A \times \langle\varphi\rangle$ contained in A.

Theorem

Let K be a subgroup of the kernel of a skew-morphism φ of a group A that is preserved by φ and is normal in A. Then φ induces a 'factor' skew-morphism φ^* on A/K defined by

$$\varphi^*(aK) = \varphi(a)K.$$

Cyclic Extensions from Skew-Morphisms

Theorem (Lucchini)

If P is a transitive permutation group of degree $n > 1$ with cyclic point-stabilizers, then $|P| \leq n(n-1)$.

Theorem (Herzog and Kaplan)

Let A be a non-trivial finite group of order n with a cyclic subgroup $\langle x \rangle$ satisfying the property $|x| \geq \sqrt{n}$. Then $\langle x \rangle$ contains a non-trivial normal subgroup of A.

Properties of Skew-Morphisms

Theorem (Khoroshevskij)

The order of every automorphism of a finite group H is less than $|H|$.

Properties of Skew-Morphisms

Theorem (Khoroshevskij)

The order of every automorphism of a finite group H is less than $|H|$.

Theorem (Conder, RJ, Tucker)

If φ is a skew morphism of the non-trivial finite group A, then the order of φ is less than $|A|$.

Properties of Skew-Morphisms

Theorem (Khoroshevskij)

The order of every automorphism of a finite group H is less than $|H|$.

Theorem (Conder, RJ, Tucker)

If φ is a skew morphism of the non-trivial finite group A, then the order of φ is less than $|A|$.

Corollary (Conder, RJ, Tucker)

Every skew morphism of a non-trivial finite group has non-trivial kernel.

Properties of Skew-Morphisms

Theorem (Khoroshevskij)

The order of every automorphism of a finite group H is less than $|H|$.

Theorem (Conder, RJ, Tucker)

If φ is a skew morphism of the non-trivial finite group A, then the order of φ is less than $|A|$.

Corollary (Conder, RJ, Tucker)

Every skew morphism of a non-trivial finite group has non-trivial kernel.

Corollary (Conder, RJ, Tucker)

Every skew morphism of a cyclic group of prime order is an automorphism.

Kernels of Skew-Morphisms

Theorem (Conder, RJ, Tucker)

If A is a finite abelian group of order greater than 2, then the kernel of every skew morphism of A has order greater than 2.

Kernels of Skew-Morphisms

Theorem (Conder, RJ, Tucker)

If A is a finite abelian group of order greater than 2, then the kernel of every skew morphism of A has order greater than 2.

Theorem (Conder, RJ, Tucker)

Let A be a finite abelian group of order greater than 2. If K is the kernel of any skew morphism of A, then every prime divisor of $|K|$ is larger than every prime that divides $|A|$ but not $|K|$.
In particular if q is the largest prime divisor of $|A|$, then the order of the kernel of every skew morphism of A is divisible by q when q is odd, or by 4 when $q = 2$.

Kernels of Skew-Morphisms

Theorem (Conder, RJ, Tucker)

If A is a finite abelian group of order greater than 2, then the kernel of every skew morphism of A has order greater than 2.

Theorem (Conder, RJ, Tucker)

Let A be a finite abelian group of order greater than 2. If K is the kernel of any skew morphism of A, then every prime divisor of $|K|$ is larger than every prime that divides $|A|$ but not $|K|$.
In particular if q is the largest prime divisor of $|A|$, then the order of the kernel of every skew morphism of A is divisible by q when q is odd, or by 4 when $q = 2$.

Corollary (Conder, RJ, Tucker)

Every skew morphism of an elementary abelian 2-group is an automorphism.

Further Results

Theorem (Conder, RJ, Tucker)

Let φ be a skew morphism of C_n. Then the order m of φ divides $n\phi(n)$. Moreover, if $gcd(m, n) = 1$ or $gcd(\phi(n), n) = 1$, then φ is an automorphism of C_n.

Further Results

Theorem (Conder, RJ, Tucker)

Let φ be a skew morphism of C_n. Then the order m of φ divides $n\phi(n)$. Moreover, if $gcd(m,n) = 1$ or $gcd(\phi(n), n) = 1$, then φ is an automorphism of C_n.

Theorem

Let A be any finite abelian group. Then every skew morphism of A is an automorphism of A if and only if A is is cyclic of order n where $n = 4$ or $\gcd(n, \phi(n)) = 1$, or A is an elementary abelian 2-group.

Further Results

Theorem (Conder, RJ, Tucker)

Let φ be a skew morphism of C_n. Then the order m of φ divides $n\phi(n)$. Moreover, if $gcd(m,n) = 1$ or $gcd(\phi(n), n) = 1$, then φ is an automorphism of C_n.

Theorem

Let A be any finite abelian group. Then every skew morphism of A is an automorphism of A if and only if A is is cyclic of order n where $n = 4$ or $\gcd(n, \phi(n)) = 1$, or A is an elementary abelian 2-group.

Classification and enumeration of the skew-morphisms of the cyclic groups C_{p^2} and C_{pq} and of $C_p \times C_p$.

Open Problem:

The set of all skew-morphisms of a finite group A is a subgroup of

$\mathbb{S}_A$

if and only if

all the skew-morphisms of A are group automorphisms of A.

Back to Maps

Definition

A **mappical regular regular representation** of a group G is a Cayley map $\mathcal{M} = CM(G, X, p)$ whose full (orientation preserving) automorphism group is equal to G_L.

Definition

A **mappical regular regular representation** of a group G is a Cayley map $\mathcal{M} = CM(G, X, p)$ whose full (orientation preserving) automorphism group is equal to G_L.

Theorem (RJ)

Every finite group other than $\mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ admits an MRR.

Back to Maps

Definition

A **mappical regular regular representation** of a group G is a Cayley map $\mathcal{M} = CM(G, X, p)$ whose full (orientation preserving) automorphism group is equal to G_L.

Theorem (RJ)

Every finite group other than $\mathbb{Z}_3$ *and* $\mathbb{Z}_2 \times \mathbb{Z}_2$ *admits an MRR.*

Theorem (Bachratý, RJ)

Let $\mathcal{M} = CM(G, X, p)$ *be a Cayley map. The stabilizer of* 1_G *in* $Aut(\mathcal{M})$ *is non-trivial if and only if there exists a skew-morphism of* G *that preserves* X.

Conjecture:

Almost all Cayley maps are MRR's.

Back to Maps

Conjecture:

Almost all Cayley maps are MRR's.

$$\lim_{n\to\infty} \frac{\#\text{ of MRR's of order } \leq n}{\#\text{ of Cayley maps of order } \leq n} = 1$$

Conjecture:

Almost all Cayley maps are MRR's.

$$\lim_{n\to\infty} \frac{\#\text{ of MRR's of order } \leq n}{\#\text{ of Cayley maps of order } \leq n} = 1$$