An exact method
for the \((r|p)\)–centroid problem
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(r|p)–centroid problem

• Input:
  J is the set of users;
  I is the set of potential facilities;
  p is the total number of facilities opened by the Leader;
  r is the total number of facilities opened by the Follower;
  w_j is the profit for servicing of the user j;
  g_{ij} is the distance between the user j and the facility i;

• Output: p facilities opened by the Leader;

• Goal: maximize the total profit for the Leader.
Example

$I = J, |I| = 9$
Leader has opened $p$ facilities. Leader’s market share is 100%.

$I=J$, $|I|=9$, $p=2$
Follower has opened \( r \) facilities.

Leader’s market share is 56%.

\[ I=J, \ |I|=9, \ p= r=2 \]
Mathematical Formulation

Leader Variables \( x_i = \begin{cases} 1, & \text{if the Leader opens facility } i, \\ 0, & \text{otherwise}, \end{cases} \)

Follower Variables \( y_i = \begin{cases} 1, & \text{if the Follower opens facility } i, \\ 0, & \text{otherwise}, \end{cases} \)

User Variables \( u_j = \begin{cases} 1, & \text{if user } j \text{ is serviced by the Leader,} \\ 0, & \text{if user } j \text{ is serviced by the Follower.} \end{cases} \)

For the given solution \( x_i, i \in I \) we define the set of facilities

\[ I_j(x) = \{ i \in I \mid g_{ij} < \min_{k \in I} (g_{kj} \mid x_k = 1) \} \]

which allows to “capture” the user \( j \) by the Follower.
Bilevel 0-1 Model

\[
\max_x \sum_{j \in J} w_j u_j^*(x, y^*)
\]

s.t. \[
\sum_{i \in I} x_i = p, \quad x_i \in \{0, 1\}, i \in I
\]

where \( u_j^*(x, y^*) \), \( y_i^* \) is the optimal solution of the Follower problem:

\[
\max_{u_j, y_i} \sum_{j \in J} w_j (1 - u_j)
\]

s.t. \[
1 - u_j \leq \sum_{i \in I} y_i, j \in J
\]

\[
\sum_{i \in I} y_i = r
\]

\( y_i, u_j \in \{0, 1\}, i \in I, j \in J \)
# Complexity Status

| (r|p)-centroid | NP-hard, S. Hakimi, 1990 |
|----------------|--------------------------|
|                | $\sum \frac{p}{2}$-hard on graph, |
|                | H. Noltemeier, J. Spoerhase, H. Wirth, 2007 |
|                | NP-hard on spider |
|                | $O(pn^4)$ on path |
|                | J. Spoerhase, H.-C. Wirth, 2008 |
| (1|p)-centroid   | $O(n^2(\log n)^2 \log W)$ on tree |
|                | NP-hard on pathwidth bounded graph |
| (1|1)-centroid   | polynomial solvable on graph and on a network |
|                | P. Hansen, M. Labbé, 1988 |
Computational Methods

- **Tabu search algorithm**, $|I| = |J| = 70, p, r \leq 3$
  S. Benati, G. Laporte, 1994
- **An alternating heuristic on the plane**, $|J| \leq 100, p, r \leq 25$
  J. Bhadury, H. A. Eiselt, J. H. Jaramillo, 2001
- **Hybrid memetic algorithm**, $|I| = |J| = 100, p = r \leq 10$
  E. Alekseeva, N. Kochetova, Y. Kochetov, A. Plyasunov, 2009
- **The partial enumeration algorithm**, $|I| \leq 50, |J| \leq 100, p, r \leq 5$
  C.M.C. Rodríguez, J.A. Moreno Pérez, 2008
- **Three MIP models**, $|I| = |J| \leq 25, r = 1, p \geq 1$ (arbitrary)
  F. Plastria, L. Vanhaverbeke, 2008
Main Results

New reformulation as Integer Linear Program

An exact algorithm

Computational experiments on the large scale instances
Notations

Let \( F \) be the set of all feasible solutions of the Follower.

For \( y \in F \) define \( I_j(y) = \{ i \in I \mid g_{ij} < \min_{k \in I} (g_{kj} \mid y_k = 1) \} \), \( j \in J \)

the set of the Leader’s facilities which allows the Leader to keep client \( j \) if the Follower will use the solution \( y \).

Introduce new variables:

\[
\begin{align*}
u_j^y &= \begin{cases} 
1, & \text{if client } j \text{ is served by the Leader when the Follower uses solution } y \\
0, & \text{if user } j \text{ is serviced by the Follower when the Follower uses solution } y 
\end{cases}
\end{align*}
\]
Integer Linear Program

\[
\begin{align*}
\max_{UB, x, u} & \quad UB \\
\text{s.t.} & \quad \sum_{i \in I} x_i = p \\
& \quad \sum_{j \in J} w_j u_j^y \geq UB, \; y \in F \\
& \quad u_j^y \leq \sum_{i \in I_j (y)} x_i, \; j \in J, \; y \in F \\
& \quad u_j^y \in \{0, 1\}, \; j \in J, \; y \in F \\
& \quad x_i \in \{0, 1\}, \; i \in I
\end{align*}
\]
Column Generation Method for \((r|p)\)-centroid problem

1. Choose an initial family \(F\)
2. Find \(UB(F)\) and \(x(F)\)
3. Solve the Follower problem and calculate \(LB(F)\)
4. If \(UB(F) = LB(F)\) then stop
5. Add \(y(F)\) in the family \(F\) go to the step 2.
The sets $I = J$, $|I| = |J|$.

The element $g_{ij}$ is an Euclidean distance between points $i \in I$ and $j \in J$, the points are randomly generated following the uniform distribution on a 7000*7000 square.

The profit $w_j$ equals one for all $j \in J$ or $w_j$, $j \in J$ is randomly generated following the uniform distribution on a $(0, 200)$ interval.

PC Pentium Intel Core 2, 1.87 GHz, RAM 2Gb, Windows XP Professional operating system, GAMS
Optimal solutions, $|I|=100$, $p=r=5$

<table>
<thead>
<tr>
<th>$w_j = 1, j \in J$</th>
<th>$w_j \in (0, 200), j \in J$</th>
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</table>
$|l|=50, p=7$

$|l|=50, r=7$
Leader’s Market Share, $|l|=50$, $w_j \in (0, 200)$
The Number of Iterations, $|F(p)|$, $|I|=50$, $p=r$
The Number of Iterations, $|F(p)|$, $|I|=50$, $p=7$
The Number of Iterations, $|F(p)|$, $|l|=50$, $r=7$
Conclusion

✓ $\sum_{2}^{P}$-hard problem has been studied
✓ A new MIP reformulation with the exp number of constraints has been suggested
✓ A new exact method has been proposed
✓ The optimal solutions for the instances with $|I|=|J|=100$ and $p=r=5$ have been found