A Hybrid Exact Method for the Discrete \((r|p)\)-centroid Problem

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Outline

1. Linear 0-1 bi-level formulation of the discrete \((r|p)\)-centroid problem
2. Reformulation and upper bounds
3. Non-classical column generation method
4. A new matheuristic: Hybrid Iterative Exact Method
5. Computation results
6. Conclusions
Discrete \((r|p)\)-centroid Problem

- **Input:**
  
  \[J\] the set of users
  
  \[I\] the set of potential facilities
  
  \[p\] the total number of facilities opened by the **Leader**
  
  \[r\] the total number of facilities opened by the **Follower**
  
  \[w_j\] the profit received from the user \(j\)
  
  \[g_{ij}\] the distance between the user \(j\) and the facility \(i\)

- **Output:** \(p\) facilities opened by the **Leader**;

- **Goal:** maximize the total profit for the **Leader**.
Example

$I = J$, $|I| = 9$
Leader has opened $p$ facilities. Leader’s market share is 100%.

$I=J, |I|=9, p=2$
Follower has opened \( r \) facilities.
Leader’s market share is 56%.

\[ I=J, \ |I|=9, \ p= r=2 \]
Leader Variables \[ x_i = \begin{cases} 1, & \text{if the Leader opens facility } i, \\ 0, & \text{otherwise,} \end{cases} \]

Follower Variables \[ y_i = \begin{cases} 1, & \text{if the Follower opens facility } i, \\ 0, & \text{otherwise,} \end{cases} \]

User Variables \[ u_j = \begin{cases} 1, & \text{if user } j \text{ is serviced by the Leader,} \\ 0, & \text{if user } j \text{ is serviced by the Follower.} \end{cases} \]

For the given solution \( x_i, i \in I \) we define the set of facilities
\[ I_j(x) = \{ i \in I \mid g_{ij} < \min_{k \in I} (g_{kj} \mid x_k = 1) \} \]
which allows to “capture” the user \( j \) by the Follower.
Bilevel 0-1 Model

\[
\begin{align*}
\max & \quad \sum_{j \in J} w_j u_j^*(x, y^*) \\
\text{s.t.} & \quad \sum_{i \in I} x_i = p, \quad x_i \in \{0,1\}, \quad i \in I
\end{align*}
\]

where \( u_j^*(x, y^*) \), \( y_i^* \) is the optimal solution of the Follower problem:

\[
\begin{align*}
\max & \quad \sum_{j \in J} w_j (1 - u_j) \\
\text{s.t.} & \quad 1 - u_j \leq \sum_{i \in I_j(x)} y_i, \quad j \in J
\end{align*}
\]

\[
\begin{align*}
\sum_{i \in I} y_i & = r \\
y_i, \quad u_j & \in \{0,1\}, \quad i \in I, \quad j \in J
\end{align*}
\]
| Complexity Status | (r|p)-centroid | NP-hard on spider |
|-------------------|---------------|-------------------|
| \( \sum_{2}^{P} \) hard on graph, | J.Spoerhase, H.-C.Wirth, 2008 |
| H. Noltemeier, J. Spoerhase, H. Wirth, 2007 | |
| NP-hard on spider | |
| \( O(pn^4) \) on path | |
| (1|p)-centroid | \( O(n^2 \log^2 n \log W) \) on tree, where \( W = \sum_{j \in J} w_j \) |
| NP-hard on pathwidth bounded graph | |
| (1|1)-centroid | \( O(n^3) \) C. M. Campos Rodríguez and J. A. Moreno Peréz, 2003 |
Computational Methods

- **Tabu search algorithm**, $|I| = |J| = 70$, $p, r \leq 3$
  S. Benati, G. Laporte, 1994

- **An alternating heuristic on the plane**, $|J| \leq 100$, $p, r \leq 25$
  J. Bhadury, H. A. Eiselt, J. H. Jaramillo, 2001

- **The partial enumeration algorithm**, $|I| \leq 50$, $|J| \leq 100$, $p, r \leq 5$
  C.M.C. Rodríguez, J.A. Moreno Pérez, 2008

- **Hybrid memetic algorithm**, $|I| = |J| = 100$, $p = r \leq 10$
  E. Alekseeva, N. Kochetova, Y. Kochetov, A. Plyasunov, 2009
Reformulation of the problem

Let $F$ be the set of all feasible solutions of the Follower.

For $y \in F$ define $I_j(y) = \{i \in I \mid g_{ij} < \min_{k \in I}(g_{kj} \mid y_k = 1)\}$, $j \in J$ the set of the Leader’s facilities which allows the Leader to keep client $j$ if the Follower will use the solution $y$.

Introduce new variables:

$$z_{jy} = \begin{cases} 
1, & \text{if user } j \text{ is serviced by the Leader when the Follower uses solution } y \\
0, & \text{if user } j \text{ is serviced by the Follower when the Follower uses solution } y 
\end{cases}$$
Large-Scale Integer Linear Program

If $F$ contains all possible Follower’s solutions then the initial problem is equivalent to the following problem:

$$\begin{align*}
\max_{W, x, u} & \quad W \\
\text{s.t.} & \quad \sum_{i \in I} x_i = p \\
& \quad \sum_{j \in J} w_j z_{jy} \geq W, \ y \in F \\
& \quad z_{jy} \leq \sum_{i \in I_j(y)} x_i, \ j \in J, \ y \in F \\
& \quad z_{jy} \in \{0, 1\}, \ j \in J, \ y \in F \\
& \quad x_i \in \{0, 1\}, \ i \in I
\end{align*}$$
Non-classical Iterative Column Generation Method

1. Choose an initial family $F$

2. Find an upper bound $W(F)$ and a solution $x(F)$

3. Solve the Follower problem find $y(F)$ and calculate $LB(F)$

4. If $W(F) = LB(F)$ then return the best found solution and stop

5. Add $y(F)$ in the family $F$ go to the step 2.
CPLEX Running Time per Iteration

![Graph showing the running time of CPLEX per iteration. The x-axis represents the iteration number, and the y-axis represents the time in minutes. The graph shows a steady increase in time as the iteration number increases.]
Feasibility Problem

Let $F$ be the subset of a set of all feasible solutions of the Follower, $W^*$ – the optimum for the initial bi-level problem

$$\sum_{j \in J} w_j z_{jy} > W^*, \; y \in F$$

$$z_{jy} \leq \sum_{i \in I_j(y)} x_i, \; j \in J, \; y \in F$$

$$\sum_{i \in I} x_i = p$$

$$0 \leq z_{jy} \leq 1, \; j \in J, \; y \in F$$

$$x_i \in \{0, 1\}, \; i \in I$$
**Hybrid Iterative Exact Method**

1. Find the best value $W^*$ by Hybrid Memetic Algorithm

2. Generate an initial family $F$

3. Find the *batch of the best leader’s solutions* by Probabilistic Tabu Search

4. Update the family

5. Solve the feasibility problem exactly

6. If the feasibility problem is infeasible then *stop*, else update *the family* and go to step 3
Computational Experiments
http://math.nsc.ru/AP/benchmarks/english.html

The sets $I = J$, $|I| = |J|

The element $g_{ij}$ is an Euclidean distance between points $i \in I$ and $j \in J$, the points are randomly generated following the uniform distribution on a 7000*7000 square.

The profit $w_j$, $j \in J$ is generated randomly following the uniform distribution on a (0,200) interval.

PC Pentium Intel Core 2, 1.87 GHz, RAM 2Gb, Windows XP Professional operating system, CPLEX
Parameters for Hybrid Exact Method.
Correlation between Batch Size and Family Size

![Bar chart showing the correlation between batch size and family size.](chart.png)
Parameters for Hybrid Exact Method: Batch Size

![Bar chart showing average batch size over iteration number]
Computational results. Optimal solutions
$|I|=|J|=100$, $p=r=5$

<table>
<thead>
<tr>
<th>$w_j = 1$, $j \in J$</th>
<th>$w_j \in (0,200)$, $j \in J$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>opt</strong></td>
<td><strong>opt</strong></td>
</tr>
<tr>
<td>47</td>
<td>4139</td>
</tr>
<tr>
<td>48</td>
<td>4822</td>
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<tr>
<td>45</td>
<td>4215</td>
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<td>47</td>
<td>4483</td>
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<td>5153</td>
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<td>4404</td>
</tr>
<tr>
<td>47</td>
<td>4700</td>
</tr>
<tr>
<td>47</td>
<td>4923</td>
</tr>
</tbody>
</table>
## Computational results

$|I|=|J|=100$, $p=r=10$, $w_j \in (0,1)$, $j \in J$

<table>
<thead>
<tr>
<th>Code</th>
<th>Lower Bound</th>
<th>Upper Bound</th>
<th>Family size</th>
<th>Time for CPLEX (minutes)</th>
<th>Time for Tabu Search (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>111</td>
<td>4361</td>
<td>5036 (10%)</td>
<td>105</td>
<td>83</td>
<td>58</td>
</tr>
<tr>
<td>211</td>
<td>5310</td>
<td>5575 (5%)</td>
<td>210</td>
<td>343</td>
<td>90</td>
</tr>
<tr>
<td>311</td>
<td>4483</td>
<td>4931 (10%)</td>
<td>121</td>
<td>188</td>
<td>85</td>
</tr>
<tr>
<td>411</td>
<td>4985</td>
<td>5234 (5%)</td>
<td>217</td>
<td>402</td>
<td>95</td>
</tr>
<tr>
<td>511</td>
<td>4876</td>
<td>5363 (10%)</td>
<td>171</td>
<td>216</td>
<td>117</td>
</tr>
<tr>
<td>611</td>
<td>4587</td>
<td>5045 (10%)</td>
<td>168</td>
<td>160</td>
<td>140</td>
</tr>
<tr>
<td>711</td>
<td>5463</td>
<td>6009 (10%)</td>
<td>118</td>
<td>150</td>
<td>63</td>
</tr>
<tr>
<td>811</td>
<td>4537</td>
<td>4990 (10%)</td>
<td>161</td>
<td>320</td>
<td>140</td>
</tr>
<tr>
<td>911</td>
<td>5302</td>
<td>5567 (5%)</td>
<td>142</td>
<td>175</td>
<td>91</td>
</tr>
<tr>
<td>1011</td>
<td>4936</td>
<td>5429 (10%)</td>
<td>160</td>
<td>180</td>
<td>120</td>
</tr>
</tbody>
</table>
Conclusions

✓ $\sum_{P}^{2}$-hard problem has been studied
✓ A new MIP reformulation with the exp number of constraints has been suggested
✓ A new hybrid exact method based on a non-classical column generation approach has been proposed
✓ Solutions with at most 10% gap for the instances with $|I|=|J|=100$ and $p=r=10$ have been found