INVERSE PROBLEMS FOR EQUATIONS WITH MEMORY
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Abstract. We give an overview of some recent results obtained by the authors together with V. M. Isakov and N. I. Kalinina for the problems of memory reconstruction. Most of the results concern the case in which the memory depends on all space variables. We prove some uniqueness and stability theorems.

Key words. Equations with memory, the Riesz equation, the Dirichlet-to-Neumann map, special exponential solutions, the Newton method

AMS subject classifications. 35R30

Introduction

In the present article, we expose some recent results obtained by the authors, V. M. Isakov, and N. I. Kalinina for problems of memory reconstruction. By the latter we mean the inverse problems of finding the function \( k(x,t) \) in a convolution term like \( \int_0^t k(x,t-\tau)u(x,\tau)\,d\tau \) in a hyperbolic or parabolic equation from some additional information. Here \( u(x,t) \) is either solution or some of its derivative, for instance, \( u_{tt} \).

The fact that many physical processes are described by integrodifferential equations was observed already by Boltzmann and Volterra. As a simplest example consider the equations of longitudinal oscillations in a viscoelastic rod \([\text{LoSu}]\):

\[
\rho(x)u_{tt} = \sigma_x + f(x,t);
\sigma(x,t) = E(x) \left( \varepsilon(x,t) - \int_{-\infty}^t h(x,t-\tau)\varepsilon(x,\tau)\,d\tau \right);
\varepsilon(x,t) = u_x.
\]

Here \( u \) is the displacement, \( \rho \) is the density, \( \varepsilon \) is the stress, \( \sigma \) is the strain, \( E \) is the momentary elasticity modulus, and \( h(x,t) \) is the relaxation kernel. The form of the second equation which connects the stress and the strain is stipulated by the supposition that \( \sigma \) depends on \( \varepsilon \) linearly and this dependence does not change with time (i.e., the material does not become obsolete). The system can be rewritten as a single equation:

\[
\rho(x)u_{tt} = \frac{\partial}{\partial x} (Eu_x) + \frac{\partial}{\partial x} \int_{-\infty}^t E(x)h(x,t-\tau)u_x(x,\tau)\,d\tau = f(x, t).
\]

The equations arising in the multidimensional case have a similar form but are even more bulky. Therefore, qualitative study of such problems starts with equations like \( u_{tt} - \Delta u + \int_0^t k\Delta u\,d\tau = f(x,t), u_{tt} - \Delta u + \int_0^t ku_{\tau\tau}\,d\tau = f(x,t) \) (the latter is obtained...
from the former by solving the Volterra integral equation \(-\Delta u + \int_0^t k \Delta u \, d\tau = f - u_{tt}\) for \(\Delta u\), or even \(u_{tt} - \Delta u + \int_0^t ku \, d\tau = f(x,t)\).

Equations with convolution terms arise also in description of other processes: heat conduction, electromagnetic wave propagation, etc.

Since the bibliography of the works devoted to problems of memory reconstruction comprises several tens of articles, the complete overview of this topic would be too long. So we only mention the most important works (to the authors’ opinion).

Apparently, the first systematic results in this field were obtained by Italian mathematicians (see [L, LP1, LS1, GL, G1]). They dealt with abstract parabolic and hyperbolic equations with memory. For example, consider the following problem: Find the pair of functions \(u(t)\) and \(h(t)\) satisfying the problem

\[
(0.1) \quad u_t - Au - \int_0^t h(t - \tau) Bu(\tau) \, d\tau = f(t), \quad t \in [0, T], \quad u(0) = u_0
\]

from the additional information \(\Phi[u(t)] = g(t)\). Here \(A\) and \(B\) are operators in a Banach space \(X\); \(u\) and \(f\) act from \([0, T]\) into \(X\); \(h\) and \(g\) are scalar functions on \([0, T]\); and \(\Phi\) is a continuous linear functional on \(X\). Under certain conditions, differentiating the equation and inverting the operator \(\frac{d}{dt} - A\) by means of the semi-group \(e^{tA}\), we obtain the system of nonlinear integral equations

\[
(0.2) \quad w = e^{tA} v_0 + A e^{tA} * (hBu_0 + h * BA^{-1}w + f'), \quad h = \chi^{-1} g'' - \chi^{-1} \Phi[w + h * BA^{-1}w + f']
\]

in the functions \(h\) and \(w = Au_t\); here \(v_0 = Au_0 + f(0)\) and \(\chi = \Phi[Bu_0]\) is assumed to be nonzero. Solving the so-obtained system by means of the contraction mapping principle, one obtains local existence and global uniqueness and stability. These results were obtained in [LS1, LS2, LP1]. Similar results were obtained for hyperbolic equations [GKL1, GKL2], the Maxwell equations [LPr, KL], the equations of viscoelasticity [LUYa], the equations of thermoviscoelasticity [G1, CG], some nonlinear equations [GL], etc. Most of the above results are reviewed in [L].

Regarding the convolution nature of the system (0.2), A. L. Bukhgeĩm [B1] proposed to use the weighted \(L_p\) spaces with the norm \(\|u\|_{p,\sigma} = \left(\int_0^T \|e^{-\sigma t}u(t)\|^p \, dt\right)^{1/p}\) for solving (0.2) and obtaining global existence theorems for problems like (0.1). J. Janno [J] employed an analogous construction for solving an inverse problem for a one-dimensional hyperbolic equation. After that global existence was obtained in various problems of memory reconstruction (see, for example, [W] and the bibliography therein).

In the case of a one-dimensional hyperbolic equation with memory on \(x > 0, -\infty < t < \infty\), L. von Woltersdorf reduced the inverse problem of memory reconstruction to the Riemann–Hilbert problem and obtained some existence results (see [W] and the bibliography therein).

Observe that in all works mentioned above the memory is independent of the space variables. In [CL, LYa], some local results were obtained for problems in which the memory depends on one space variable.

Another problem with space-dependent memory was considered by D. K. Dur-
diev [D2]: Find the pair \((u, k)\) in the problem
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u &= \int_0^t k(x', \tau) u(x, t - \tau) \, d\tau, \quad (x, t) = (x', x_n, t) \in \mathbb{R}^{n+1}, \ x_n > 0; \\
u|_{t<0} &\equiv 0; \quad u_{x_n}|_{x_n=0} = \delta'(t) + h(x', t)\theta(t), \quad \ (x', t) \in \mathbb{R}^n
\end{align*}
\]
from the additional information \(u|_{x_n=0} = g(x', t), \ (x', t) \in \mathbb{R}^n\). Considering the structure of the solution to (0.3), one can show that \(g(x', t) = \delta(t) + f(x', t)\theta(t)\). On assuming that the given functions \(h\) and \(f\) and the sought function \(k\) are analytic in \(x'\), the author reduces the inverse problem to a system of nonlinear integrodifferential equations. Solving this system, one can obtain local existence and global uniqueness results.

In the present article, we give a review of some methods proposed by the authors for solving inverse problems of memory reconstruction in the case when the memory depends on all space variables. The main idea is reduction of the inverse problems to a family of stationary problems by means of the Fourier transform. In the first section, we consider the problem of simultaneous determination of the absorption and the memory in a hyperbolic equation in the whole space from the backward scattering data. In the second and third sections, we study the problem of finding the memory from the so-called Dirichlet-to-Neumann map for hyperbolic and parabolic equations. At last, in the fourth section, which stands somewhat apart from the first three, we present some results on global convergence of the Newton method in inverse problems like (0.1).

§ 1. Determination of the Memory from the Scattering Data

1.1. Statement of the problem and the main result. We consider the Cauchy problem
\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \Delta u + \omega(x)u_t + \int_0^t k(x, t - \tau)u_{\tau\tau}(x, \tau) \, d\tau &= \delta(x - x_0, t), \\
u|_{t<0} &\equiv 0
\end{align*}
\]
in \(\mathbb{R}^{n+1}, \ n = 2\nu + 1, \ \nu = 1, 2, \ldots\). We suppose that the absorption \(\omega(x)\) and the memory \(k(x, t)\) are localized in some bounded domain \(\Omega \subset \mathbb{R}^n\), i.e., \(\omega(x) = 0\) and \(k(x, t) = 0\) for \(x \not\in \Omega\). Moreover, we suppose that \(k\) belongs to the space \(C^2(\overline{\Omega} \times [0, \infty))\) and decreases rapidly as \(t \to \infty\) in the sense that
\[
\sup_{x \in \Omega}|k(x, t)t^n| \to 0, \quad t \to \infty.
\]
Assume that the point \(x_0\) varies in a domain \(\Omega_0 \subset \mathbb{R}^n\) such that \(\Omega \cap \Omega_0 = \emptyset\). The shape and the size of \(\Omega_0\) are inessential.

The Cauchy problem (1.1), (1.2) has a unique generalized solution \(u(x, t, x_0)\). Moreover, the solution has the form
\[
u(x, t) = E(x - x_0, t) + v(x, t, x_0),
\]
where \(E(x, t)\) is the fundamental solution of the wave operator which vanishes for \(t < 0\) and \(v(x, t, x_0)\) is the scattered wave.
We consider the inverse problem of determination of $\omega(x)$ and $k(x,t)$ from the scattering data $v(x,t,x)$, $x \in \Omega_0$, $t \in \mathbb{R}$. The main result of this section is the following uniqueness theorem:

**Theorem 1.1.** Suppose that the functions $\omega(x)$ and $k(x,t)$ satisfy the above requirements and the scattered wave $v(x,t,x_0)$ rapidly decreases as $t \to \infty$ in the following sense:

\begin{equation}
\sup_{x \in \Omega \cup \Omega_0} |v(x,t,x_0)|^m \to 0, \quad t \to \infty.
\end{equation}

Let $k(x,t) = k_0(x,t) + k_1(x,t)$, where $k_0$ is a known function which satisfies the same constraints as $k(x,t)$ and $k_1(x,t)$ is an unknown function vanishing for $t \geq T$, with $T$ some positive number; i.e., $k(x,t) = 0$, $t \geq T$. Then $\omega(x)$ and $k(x,t)$ are uniquely determined by the scattering data $v(x,t,x)$, $x \in \Omega_0$, $t \in \mathbb{R}$.

**1.2. Proof of Theorem 1.1.** Inserting (1.3) into (1.1), we obtain the following Cauchy problem in $v(x,t,x_0)$:

\begin{equation}
v_{tt} - \Delta v = -\partial_t (\omega(E + v)) - \partial_t^2 \int_{-\infty}^{\infty} k(x,t-\tau)(E(x-x_0,\tau) + v(x,\tau,x_0)) \, d\tau, \quad v|_{t<0} = 0,
\end{equation}

which is equivalent to the integral equation

\begin{equation}
v = -E \star \left[ \partial_t (\omega(E + v)) + \partial_t^2 \int_{-\infty}^{\infty} k(x,t-\tau)(E(x-x_0,\tau) + v(x,\tau,x_0)) \, d\tau \right].
\end{equation}

Here we extend the function $k(x,t)$ by zero to $t < 0$ and $\ast$ denotes the convolution in the variables $x$ and $t$.

For a function $w(x,t)$, $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, define the Fourier transform in time as follows:

\[ \hat{w}(x,\lambda) := \int_{-\infty}^{\infty} e^{-it\lambda} w(x,t) \, dt. \]

Before applying the Fourier transform to the equation (1.5), we observe that

\[ \hat{E}(x,\lambda) = \sum_{l=0}^{\nu-1} C_l \frac{(-i\lambda)^l e^{-i\lambda|x|}}{|x|^{2l-1-t}} \]

with some absolute constants $C_l$ (one can obtain this, using the form of $E$ (see [H])).

Now, using the formula

\[ (u * v) = \int \hat{u}(x-y,\lambda) \hat{v}(y,\lambda) \, dy, \]

apply the Fourier transform in time to (1.5):

\begin{equation}
\hat{v}(x,\lambda,x_0) = -\int_{\Omega} \hat{E}(x-y,\lambda) \left( i\lambda \omega(y) + (i\lambda)^2 \hat{k}(y,\lambda) \right) \times \left( \hat{E}(y-x_0,\lambda) + \hat{v}(y,\lambda,x_0) \right) \, dy.
\end{equation}
where $\Omega = 0$.

By the conditions of the theorem, the Fourier transform $\hat{v}(x,\lambda, t)$ of $v(x,t)$ is infinitely differentiable with respect to $\lambda$ and we have the expansion

$$\hat{v}(x,\lambda, t) = \sum_{r=0}^{N} v_r(x,y) (i\lambda)^r + o(\lambda^N),$$

with some $v_r(x,y)$. Similarly,

$$\hat{k}(x,\lambda, t) = \sum_{r=0}^{N} k_r(x)(i\lambda)^r + o(\lambda^N).$$

Denote $\mu = i\lambda$ and $k_{-1}(x) := \omega(x)$. Inserting the above expansions into (1.6), we obtain

$$\sum_{j=0}^{N} v_j(x, t) \mu^j + o(\mu^N)$$

$$= - \int_{\Omega} \left( \sum_{l=0}^{\nu-1} \sum_{p=0}^{\infty} C_l \frac{(-\mu)^p |x-y|^p}{p!} \right) \left( \sum_{q=1}^{N} k_{q-2}(y) (i\lambda)^q + o(\mu^N) \right)$$

$$\times \left( \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} C_m \frac{(-\mu)^{r+m}|y-x_0|^{r+m-2\nu}}{r!} + \sum_{r=0}^{N} v_r(y,x_0) \mu^r + o(\mu^N) \right) dy$$

Equating the coefficients of $\mu^0$ we find that $v_0(x, x_0) = 0$. By induction, equating the coefficients of $\mu^j, j \geq 1$, we can easily establish that

$$v_j(x, x_0) = -C_2^2 \int_{\Omega} \frac{k_{j-2}(y)}{|x-y|^{n-2\alpha}} dy + \cdots,$$

where the ellipses stands for the terms (if any) that contain $k_r(y), r < j - 2$, and $v_r(x, x_0), r < j$.

Now, we need the following assertion proven by M. Riesz:

**Theorem 1.2** [R]. Let $\alpha$ be a real such that $\alpha \neq 2 + 2m$ and $\alpha \neq n + 2m$, $m = 0, 1, 2, \ldots$. If $b \in L_1(\Omega)$, then the equality

$$\int_{\Omega} \frac{b(y)}{|x-y|^{n-\alpha}} dy = 0, \quad x \in \Omega_0,$$

where $\Omega_0$ is a domain in $\mathbb{R}^n, \Omega \cap \Omega_0 = \emptyset$, implies that $b(y) = 0, y \in \Omega$.

Now, the process of determination of $k_j(x)$ from $v_j(x, x), x \in \Omega$, proceeds as follows: Applying Theorem 1.2 with $\alpha = 4 - n$ to (1.7) with $j = 1$ and $x = x_0$, we uniquely determine $k_{-1}(x)$ from $v_1(x, x)$. Afterwards, using (1.7) with different $x$ and $x_0$, we find $v_1(x, x_0)$. Having found $k_{-1}(x)$ and $v_1(x, x_0)$, we can apply again Theorem 1.2 to (1.7) with $j = 2$, to determine $k_0(x)$. Obviously, the process can be continued.
Thus, \( \omega(x) \equiv k_t(x) \) and the coefficients \( k_j(x), j \geq 0 \), are uniquely determined by the coefficients \( v_j(x, t) \). Recall that \( k(x, t) = k_0(x, t) + k_1(x, t) \). Here \( k_0(x, t) \) is a known function rapidly decreasing in time, thereby its Fourier transform \( \hat{k}_0(x, \lambda) \) is infinitely differentiable with respect to \( \lambda \) and for every \( N \in \mathbb{N} \) we have the expansion

\[
\hat{k}_0(x, \lambda) = \sum_{j=0}^{N} k_{0j}(x)(i\lambda)^j + o(\lambda^N).
\]

As for the function \( k_1(x, t) \), it is compactly supported in time and its Fourier transform \( \hat{k}_1(x, \lambda) \) is analytic in \( \lambda \), i.e.,

\[
\hat{k}_1(x, \lambda) = \sum_{j=0}^{\infty} k_{1j}(x)(i\lambda)^j.
\]

Obviously, \( k_{1j}(x) = k_j(x) - k_{0j}(x) \). Since \( k_{0j}(x) \) are known and \( k_j(x) \) are uniquely determined by the scattering data, the coefficients \( k_{1j}(x) \) and hence the Fourier transform \( \hat{k}_1(x, \lambda) \) and thereby the function \( k_1(x, t) \) are uniquely determined by the scattering data. Theorem 1.1 is proven.

In Theorem 1.1 we require that \( v(x, t, x_0) \) be rapidly decreasing in time. For \( n = 3 \), it was proven in [BD] that this condition is satisfied if \( \omega(x) = k(x, 0) \) and the memory \( \hat{k}(x, t) \) is small in a sense.

**Lemma 1.3.** If \( \omega(x) = k(x, 0) \), \( |k_{tt}(x, t)| \leq c(x)e^{-\beta t} \), and

\[
\sup_{x \in \Omega} \int_{\Omega} \left( |k_t(y, 0)| + c(y)\beta^{-1}\right) \frac{e^{\beta|x-y|}}{4\pi|x-y|} \, dy < 1,
\]

then \( v(x, t, x_0) \) rapidly decreases as \( t \to \infty \) in the sense of (1.4).

The idea of the proof is to consider (1.5) as the operator equation \( v = Kv + F \) in the weighted space \( C_{\gamma} \) of continuous functions \( w(x, t) \) on \( \overline{\Omega} \times (0, \infty) \) with the norm

\[
\|w\|_{\gamma} = \sup_{x \in \Omega_{\gamma}} |w(x, t)e^{\gamma t}|, \quad \gamma > 0.
\]

Under the conditions of the lemma, we can show that \( F \in C_{\gamma} \) and the norm of the operator \( K : C_{\gamma} \rightarrow C_{\gamma} \) is less than unity. Hence, the solution \( v \) to the operator equation as well belongs to \( C_{\gamma} \). The extension of \( v \) to the whole space (easily obtained from (1.5)) obviously satisfies the condition (1.4).

**1.3. Remarks.** The method of reducing the inverse problems for identifying the coefficients to solving the Riesz equation was first employed by M. M. Lavrent’ev in [La1, La2]. Later A. L. Bukhgeîm applied this method to finding the coefficient \( b(x) \) supported in a bounded domain in the equation

\[
\Delta v + \lambda^2(1 + b(x))v = \delta(x - x_0)
\]

from the backward scattering data [B2]. Our problem of memory reconstruction is closely connected with the problem for (1.8), since, applying the Fourier transform in time to (1.1), we obtain

\[
\Delta \hat{u} + \lambda^2(1 - \hat{k}(x, \lambda))\hat{u} - \lambda\omega(x)\hat{u} = \delta(x - x_0)
\]
which is nothing but (1.8) with $b$ depending on the parameter $\lambda$ and the extra summand $\lambda\omega(x)\bar{u}$. In the case of $n = 3$ and $\omega(x) = 0$ Theorem 1.1 was proven in [BD].

§ 2. Determination of the Memory from the Dirichlet-to-Neumann Map (the Hyperbolic Case)

2.1. Statement of the problem and the main result. Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with smooth boundary $\partial\Omega$. Consider the boundary value problem

\begin{align}
(2.1) & \quad u_{tt} - \Delta u - \int_{-\infty}^{t} k(x, t - \tau)u(x, \tau) d\tau = 0, \quad x \in \Omega, \ t \in \mathbb{R}, \\
(2.2) & \quad u|_{\partial\Omega \times \mathbb{R}} = g(x, t), \quad x \in \partial\Omega, \ t \in \mathbb{R}.
\end{align}

Here $k(x, t)$ is a smooth function given for $x \in \Omega$, $t \geq 0$. We suppose $k(x, t)$ to be extended by zero to $t < 0$. To state the solvability theorem for the problem (2.1), (2.2), we introduce the weighted spaces $H^s_\gamma(\Omega \times \mathbb{R})$ and $H^s_\gamma(\partial\Omega \times \mathbb{R})$ that are constituted by the functions $u$ such that $e^{-\gamma t}u \in H^s(\Omega \times \mathbb{R})$ and $e^{-\gamma t}u \in H^s(\partial\Omega \times \mathbb{R})$ respectively. Here $H^s(\Omega \times \mathbb{R})$ and $H^s(\partial\Omega \times \mathbb{R})$ are the standard Sobolev spaces. We denote the norms in the weighted spaces by $\| \cdot \|_{\gamma, s}$ and $\langle \cdot \rangle_{\gamma, s}$.

We now state the solvability theorem for the direct problem:

**Theorem 2.1** [BDI]. Let $K > 0$. There are positive constants $\gamma_0$ and $C$ depending only on $\Omega$ and $K$ and such that for arbitrary $\gamma \geq \gamma_0$, $k \in C(\overline{\Omega} \times [0, \infty))$, $\|k\|_C \leq K$, and $g \in H^2_\gamma(\partial\Omega \times \mathbb{R})$ there is a unique solution $u \in H^2_\gamma(\Omega \times \mathbb{R})$ to the problem (2.1), (2.2); moreover, the following estimate holds:

$$\gamma\|u\|^2_{\gamma, 2} + \langle \partial_\nu u \rangle^2_{\gamma, 0} \leq \frac{C}{\gamma}(g)^2_{\gamma, 2}$$

(here $\partial_\nu$ stands for the normal derivative).

One can derive Theorem 2.1 from the well-known results on solvability of general boundary value problems for hyperbolic equations (see, for instance, [S, VG]).

By Theorem 2.1, we can define the Dirichlet-to-Neumann map $H : H^2_\gamma(\partial\Omega \times \mathbb{R}) \to H^0_\gamma(\partial\Omega \times \mathbb{R})$ by the rule $Hg := \partial_\nu u|_{\partial\Omega \times \mathbb{R}}$, where $u$ is the solution to the problem (2.1), (2.2).

The inverse problem is to find $k(x, t)$ from the given Dirichlet-to-Neumann map. We prove a conditional stability estimate for this problem:

**Theorem 2.2.** Suppose that $k_j \in C^4(\overline{\Omega} \times [0, \infty))$, $j = 1, 2$, are two functions such that $\|k_j\|_{C^5} \leq K$, with some positive constant $K$ (which determines the correctness class). There is $\gamma_0 > 0$ depending only on $\Omega$ and $K$ such that, for every $\gamma \geq \gamma_0$, the corresponding Dirichlet-to-Neumann maps $H_j$ are defined as operators from $H^2_\gamma(\partial\Omega \times \mathbb{R})$ to $H^0_\gamma(\partial\Omega \times \mathbb{R})$ and the following estimate holds:

$$\|k_1 - k_2\|_{\gamma, 0} \leq \omega(\|H_1 - H_2\|),$$

where

$$\omega(\varepsilon) \sim C(\log \varepsilon^{-1})^{-1/(2n+3)}, \quad \varepsilon \to 0,$$

with $C$ depending only on $\Omega$, $K$, and $\gamma$. 

...
Corollary 2.3. The assertion of Theorem 2.2 remains valid if we restrict the domain of $H^1_0(\Omega)$ to $\hat{H}^2(\partial\Omega \times \mathbb{R})$ that is the completion of the set of functions in $C^\infty_0(\partial\Omega \times \mathbb{R})$ supported in $\partial\Omega \times (0, \infty)$ in the norm $(\cdot)_{\gamma,2}$.

Observe that the solution to (2.1), (2.2) with $g \in \hat{H}^2(\partial\Omega \times \mathbb{R})$ vanishes for $t < 0$. So Corollary 2.3 means that we have the same result for the initial-boundary value problem for (2.1) with the zero initial condition and the boundary condition (2.2).

Proof. We can work with functions in $C^\infty_0(\partial\Omega \times \mathbb{R})$, since this space is dense in $\hat{H}^2(\partial\Omega \times \mathbb{R})$. Take $g \in C^\infty_0(\partial\Omega \times \mathbb{R})$. Since $g$ has compact support, there is $T \in \mathbb{R}$ such that $g(x, t) = 0$ for $t < -T$. Put $\tilde{g}(x, t) := g(x, t - T)$ and observe that $\tilde{g}(x, t) = 0$ for $t < 0$. Let $u$ be the solution to (2.1), (2.2). It is easily verified that $\tilde{u}(x, t) = u(x, t - T)$ is the solution to (2.1), (2.2) with $\tilde{g}$ substituted for $g$. We have $(H\tilde{g})_{\gamma,0} = (\partial_{\nu}\tilde{u})_{\gamma,0} = e^{-\gamma T}(\partial_{\nu}u)_{\gamma,0} = e^{-\gamma T}(Hg)_{\gamma,0}$ and $(\tilde{g})_{\gamma,2} = e^{-\gamma T}(g)_{\gamma,2}$. Hence the norm of the operator $H$ calculated over the space of functions $g \in C^\infty_0(\partial\Omega \times \mathbb{R})$ such that $g(x, t) = 0$ for $t < 0$ is not less than that calculated over $\hat{H}^2(\partial\Omega \times \mathbb{R})$.

2.2. Auxiliary propositions. First we suppose that $\gamma_0$ is the one given by Theorem 2.1. In the process of the proof we possibly increase this value; moreover, we always assume that $\gamma \geq \gamma_0$.

For functions $u(x, t)$ such that $ue^{-\gamma t} \in L_2(\Omega \times \mathbb{R})$ (or $ue^{-\gamma t} \in L_2(\partial\Omega \times \mathbb{R})$) we introduce the Fourier–Laplace transform in time as follows:

$$(Fu)(x, \theta) = \hat{u}(x, \theta) := \frac{1}{\sqrt{2\pi}} \int e^{-i\theta t} u(x, t) \, dt,$$

where $\theta = \sigma - i\gamma, \sigma \in \mathbb{R}$. Observe that the Parseval identity holds:

$$(u(x, t))_{\gamma,0} = \|\hat{u}(x, \sigma - i\gamma)\|_{L_2(\partial\Omega \times \mathbb{R})},$$

where $\hat{u}$ is considered as a function of $x$ and $\sigma$.

Applying the Fourier–Laplace transform to the problem (2.1), (2.2), we obtain

$$\Delta \hat{u}(x, \theta) + (\partial_{\nu}^2 + \theta^2) \hat{u}(x, \theta) = 0,$$

where $\hat{u}(x, \theta)|_{\partial\Omega \times \mathbb{R}} = \hat{g}(x, \theta)$.

Thus, $\hat{u}(x, \theta)$ is a solution to the family of the stationary problems (2.4), (2.5) that depends on the parameter $\theta = \sigma - i\gamma, \sigma \in \mathbb{R}$.

Let $q_j(x) \in C^0(\Omega), j = 1, 2$. Consider the following equation in the domain $\Omega$:

$$\Delta v_j(x) + q_j(x)v_j(x) = 0.$$  

Suppose that the Dirichlet problem for (2.4) in $\Omega$ has at most one solution. For each $q_j(x)$ we define the so-called (stationary) Dirichlet-to-Neumann map $\Lambda_j$ as follows:

$$\Lambda_j g := \partial_{\nu} v_j|_{\partial\Omega},$$

where $v_j$ is the solution to (2.6) with the condition $v|_{\partial\Omega} = g$. The operator $\Lambda_j$ is a bounded operator from $H^1(\partial\Omega)$ into $H^0(\partial\Omega)$ which ensues from the general results on the solvability of elliptic equations (see, for example, [LM]).
LEMMA 2.4. The following identity is valid for sufficiently smooth solutions $v_j$ to (2.6), $j = 1, 2$:

\begin{equation}
\int_\Omega (q_2 - q_1) v_1 v_2 \, dx = \int_{\partial \Omega} v_1 (\Lambda_1 - \Lambda_2) v_2 \, dS.
\end{equation}

Here and in the sequel the notation like $\Lambda_1 v_1$ means that we act by the operator $\Lambda_1$ on the trace $v_1|_{\partial \Omega}$. In the case of real-valued potentials $q_j(x)$ this identity was proven in [A] and the case of general elliptic operators was considered in [I1] (see also [BDI]).

Consider the equation

\begin{equation}
(\Delta + \theta^2) v(x) + q(x) v(x) = 0.
\end{equation}

To prove Theorem 2.2, we need some generalization of one theorem of J. Sylvester and G. Uhlmann [SU] (see also [I2]).

THEOREM 2.5. Let $q(x) \in C^n(\Omega)$ with a natural $s$. There are constants $C_1$ and $C_2$ depending only on $\Omega$ and $s$ such that, for every $\zeta \in \mathbb{C}^n$ satisfying the conditions $\zeta \cdot \zeta + \theta^2 = 0$, $|\Im \zeta| \geq \sqrt{2}$, and $|\zeta| \geq C_1 \|q\|_{C^s(\Omega)}$, the equation (2.8) has a solution of the form

\begin{equation}
v(x, \zeta) = e^{\zeta \cdot x} (1 + w(x, \zeta)),
\end{equation}

where $w(x, \zeta)$ satisfies the estimate

\begin{equation}
\|w(\cdot, \zeta)\|_{H^s(\Omega)} \leq \frac{C_2}{|\zeta|} \|q\|_{C^s(\Omega)}.
\end{equation}

The constants $C_1$ and $C_2$ depend only on the domain $\Omega$ and $s$.

The dot in the statement of the theorem stands for the “inner product without complex conjugation,” i.e., $x \cdot y = x_1 y_1 + \cdots + x_n y_n$. Theorem 2.4 was proven in [I2] for $s = 0$. The case of an arbitrary $s$ is considered by analogy (see [BDI]).

Below, the letter $C$ with subscripts denotes constants that depend only on the quantities indicated in the subscript. Constants with the same subscript may differ.

COROLLARY 2.6. Under the conditions of the last theorem, we have

\begin{equation}
\|v(\cdot, \zeta)\|_{H^s(\Omega)} \leq C_{1s} e^{C_0|\zeta|}.
\end{equation}

2.3. Proof of Theorem 2.2. Suppose that $k_j(x, t)$, $j = 1, 2$, are functions satisfying the conditions of Theorem 2.2. We turn to the equation (2.4) with $k = k_j$. We denote the solution to (2.4) by $\hat{u}_j(x, t)$. Denote the corresponding Dirichlet-to-Neumann maps by $\Lambda^j_\Omega$.

By Lemma 2.4, we have the identity

\begin{equation}
\int_\Omega (\hat{k}_2(x, \theta) - \hat{k}_1(x, \theta)) \hat{u}_1(x, \theta) \hat{u}_2(x, \theta) \, dx = \int_{\partial \Omega} \hat{u}_1(x, \theta) (\Lambda^1_{\Omega} - \Lambda^2_{\Omega}) \hat{u}_2(x, \theta) \, dS.
\end{equation}

By Theorem 2.5, the equation (2.4) with $k = k_j$ possesses a solution

\begin{equation}
\hat{u}_j(x, \zeta_j, \theta) = e^{\zeta_j \cdot x} (1 + w_j(x, \zeta_j, \theta)),
\end{equation}

where $\zeta_j \in \mathbb{C}^n$, $\zeta_j \cdot \zeta_j + \theta^2 = 0$, $|\zeta_j| \geq C_1 \|k_j(\cdot, \theta)\|_{C^3(\Omega)}$, $|\Im \zeta_j| \geq \sqrt{2}$, and $w_j$ satisfies the estimate

\begin{equation}
\|w_j(\cdot, \zeta_j, \theta)\|_{H^s(\Omega)} \leq \frac{C_2}{|\zeta_j|} \|k_j(\cdot, \theta)\|_{C^3(\Omega)}.
\end{equation}
Inserting the special solutions $\hat{u}_j$ into the identity, we obtain

\begin{equation}
(2.12) \quad \int_{\Omega} \left( \hat{k}_2(x, \theta) - \hat{k}_1(x, \theta) \right) e^{i(\zeta_1 + \zeta_2) \cdot x} dx = \int_{\partial \Omega} \hat{u}_1(x, \zeta_1, \theta) (\Lambda_1^\theta - \Lambda_2^\theta) \hat{u}_2(x, \zeta_2, \theta) dS + \int_{\Omega} \left( \hat{k}_1(x, \theta) - \hat{k}_2(x, \theta) \right) e^{i(\zeta_1 + \zeta_2) \cdot x} \times \left( w_1(x, \zeta_1, \theta) w_2(x, \zeta_2, \theta) + w_1(x, \zeta_1, \theta) + w_2(x, \zeta_2, \theta) \right) dx.
\end{equation}

We now turn to choosing $\zeta_j$. Take an arbitrary $\xi \in \mathbb{R}^n$. Find $\zeta$ so that

\begin{equation}
(2.13) \quad \zeta_1 + \zeta_2 = -i\xi, \quad \zeta_1 \cdot \zeta_j + \theta^2 = 0, \quad |\zeta_j| \geq C_1 \|\hat{k}_j(\cdot, \theta)\|_{C^1(\Omega)}, \quad |3\zeta_j| \geq \sqrt{2}.
\end{equation}

We seek $\zeta_j$ in the form $\zeta_j = -i\xi/2 + (1)^j(i\mu + \lambda)$, with $\mu, \lambda \in \mathbb{R}^n$ and $\lambda \cdot \mu = \xi \cdot \lambda = 0$. The first condition in (2.13) is obviously satisfied, whereas the second leads to the system

\begin{equation}
(2.14) \quad \frac{\xi^2}{4} + \mu^2 - \lambda^2 = (\sigma^2 - \gamma^2) \quad \lambda \cdot \mu = \sigma \gamma
\end{equation}

As a solution, we take $\mu$ and $\lambda$ in $\mathbb{R}^n$ such that $\mu^2 = \sigma^2 + r^2$ and $\lambda^2 = \xi^2/4 + \gamma^2 + r^2$, $r \geq \sqrt{2}$. This is possible, since for this choice of $\mu^2$ and $\lambda^2$ the first equation becomes the identity and the second can be fulfilled by varying the angle between $\mu$ and $\lambda$, for $|\mu| \geq |\sigma|$ and $|\lambda| \geq \gamma$. We can fulfill the last two conditions in (2.13) by choosing $\gamma_0$ sufficiently large and recalling that $r \geq \sqrt{2}$.

Introduce the notation

\begin{equation}
(2.15) \quad \|\hat{k}(\xi, \theta)\| \leq C_\Omega e^{C_\Omega |\xi|} \| (\Lambda_1^\theta - \Lambda_2^\theta) \hat{u}_2(x, \zeta_2, \theta) \|_{L_2(\partial \Omega)} + \frac{C_{\Omega, K, \gamma}}{|\xi|}.
\end{equation}

Here we have used Corollary 2.6, the trace theorem, and the estimate (2.11). (We suppose that $|\zeta_j| > 1$ which is achieved by the choice $\gamma_0 \geq 1$.)

Introduce the function

\begin{equation}
\hat{g}(x, \zeta_2, \theta) := \hat{u}_2(x, \zeta_2, \theta) e^{-i\xi \cdot x}, \quad x \in \partial \Omega,
\end{equation}

where the constant $C_\Omega$ is taken from Corollary 2.6. From the same Corollary 2.6 and the trace theorem we obtain

\begin{equation}
\int_R \|\hat{g}(\cdot, \zeta_2, \theta)\|_{H^2(\partial \Omega)}^2 (1 + \sigma^2)^{3/2} d\sigma \leq \int_R C_\Omega \frac{1}{1 + \sigma^2} d\sigma = C_\Omega < \infty.
\end{equation}

Hence, the inverse Fourier transform

\begin{equation}
\hat{g}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{g}(x, \zeta_2, \sigma - i\gamma) e^{it(\sigma - i\gamma)} d\sigma
\end{equation}

belongs to $H^2_\gamma(\partial \Omega \times \mathbb{R})$ and $\|\hat{g}\|_{H^2_\gamma(\partial \Omega \times \mathbb{R})} \leq C_\Omega$. 


Integrate (2.15) with the square over \((\sigma, \xi) \in \mathbb{R}^{n+1} \mid \sigma^2 + \xi^2 \leq \rho^2\), where \(\rho > 0\), and observe that \(|\xi|^2 = \frac{\xi^2}{2} + \sigma^2 + \gamma^2 + 2r^2\), to obtain

\[
(2.16) \quad \int_{\sigma^2 + \xi^2 \leq \rho^2} |\tilde{k}(\xi, \theta)|^2 \, d\xi \, d\sigma \\
\leq C_{\Omega,K} e^{C_{\Omega} \sqrt{\rho^2 + \gamma^2 + 2r^2}} (1 + \rho^2)^3 \int_{\sigma^2 + \xi^2 \leq \rho^2} \|H_{\Omega}^2 - H_{\Omega}^1\|_{L_2(\partial \Omega)} \, d\sigma \, d\xi \\
+ C_{\Omega,K} \rho^{n+1} r^{2}. 
\]

Continue the estimate (2.16), using the identity \(F(H_1 - H_2)g = (H_{\Omega}^1 - H_{\Omega}^2)\tilde{g}\):

\[
\int_{\sigma^2 + \xi^2 \leq \rho^2} |\tilde{k}(\xi, \theta)|^2 \, d\xi \, d\sigma \\
\leq C_{\Omega,K,\gamma} e^{C_{\Omega} \sqrt{\rho^2 + 2r^2}} (1 + \rho^2)^3 \rho^n \int_{\sigma \in \mathbb{R}} \|F(H_1 - H_2)g(x, \theta)\|_{L^2(\partial \Omega)} \, d\sigma \\\n+ C_{\Omega,K} \rho^{n+1} \frac{r^2}{2}. 
\]

Now, applying the Parseval identity (2.3) and the estimate \(\langle \hat{g} \rangle_{\gamma,2} \leq C_{\Omega}\), we find that

\[
(2.17) \quad \int_{\sigma^2 + \xi^2 \leq \rho^2} |\tilde{k}(\xi, \theta)|^2 \, d\xi \, d\sigma \leq C_{\Omega,K,\gamma,n} e^{C_{\Omega} \sqrt{\rho^2 + 2r^2}} \|H_1 - H_2\|^2 + C_{\Omega,K} \rho^{n+1} \frac{r^2}{2}. 
\]

Since \(\|k\|_{C^2([0, \infty) \times \mathbb{R}^{n+1})} \leq 2K\), the function \(e^{-\gamma t}k(x, t)\) extended by zero to the whole \(\mathbb{R}^{n+1}\) belongs to \(H^{1/4}(\mathbb{R}^{n+1})\) and, consequently,

\[
\int_{\mathbb{R}^{n+1}} |\tilde{k}(\xi, \theta)|^2 (1 + |\xi|^2 + |\sigma|^2)^{1/4} \, d\xi \, d\sigma \leq C_{\Omega} K, 
\]

whence

\[
(2.18) \quad \int_{\sigma^2 + \xi^2 \geq \rho^2} |\tilde{k}(\xi, \theta)|^2 \, d\xi \, d\sigma \leq \frac{C_{\Omega} K}{(1 + \rho^2)^{1/4}} \leq \frac{C_{\Omega} K}{\rho^{1/2}}. 
\]

Combining the estimates (2.17) and (2.18), we obtain

\[
\int_{\mathbb{R}^{n+1}} |\tilde{k}(\xi, \theta)|^2 \, d\xi \, d\sigma \leq C_{\Omega,K,\gamma,n} \left( e^{C_{\Omega} \sqrt{\rho^2 + 2r^2}} \|H_1 - H_2\|^2 + \frac{\rho^{n+1}}{r^2} + \frac{1}{\rho^{1/2}} \right). 
\]

Putting \(r = \rho^{(2n+3)/4}\), we estimate the expression in the parentheses by the quantity

\[
e^{C_{\Omega} \rho^{n/(2+3/4)}} \|H_1 - H_2\|^2 + \frac{2}{\rho^{1/2}},
\]

which behaves like

\[
e^{C_{\Omega} \rho^{n/(2+3/4)}} \|H_1 - H_2\|^2 + \frac{2}{\rho^{1/2}}
\]

at large \(\rho\). Taking the square of both sides of our estimate, for large \(\rho\), we obtain

\[
\|\tilde{k}\|_{L_2(\mathbb{R}^{n+1})} \leq C_{\Omega,K,\gamma,n} \left( e^{C_{\Omega} \rho^{n/(2+3/4)}} \|H_1 - H_2\| + \frac{1}{\rho^{1/4}} \right). 
\]
Now appealing, for example, to Corollary 2.1 of [B2], we arrive at the estimate of Theorem 2.2.

Observe that one can obtain a better estimate with \( n + 2 + \varepsilon \) instead of \( 2n + 3 \), where \( \varepsilon > 0 \) is arbitrary, by using the class \( H^{1/2-\varepsilon} \) instead of \( H^{1/4} \).

2.4. Remarks. One of the main results in the field of problems of finding coefficients of PDE from the Dirichlet-to-Neumann map is existence of special solutions (see Theorem 2.4). First this result was proven for the equation \( \Delta u + qu = 0 \) by J. Sylvester and G. Uhlmann [SU]. Later their proof was modified by V. M. Isakov [I2].

The problem of finding the potential \( q \) in the hyperbolic equation

\[
\frac{\partial^2 u}{\partial t^2} - \Delta u + qu = 0
\]

from the Dirichlet-to-Neumann map was first considered by Rakesh and W. W. Symes [RS] who used the method of beam solutions. For an extensive review of this topic see [U, I1]. Theorem 2.2 was proven in [BDI].

§ 3. Determination of the Memory from the Dirichlet-to-Neumann Map (The Parabolic Case)

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n, n \geq 3 \), with smooth boundary \( \partial \Omega \). Consider the boundary value problem

\[
(3.1) \quad u_t - \Delta u - \int_{-\infty}^t k(x, t - \tau)u(x, \tau)\,d\tau = 0, \quad x \in \Omega, \ t \in \mathbb{R},
\]

\[
(3.2) \quad u|_{\partial \Omega \times \mathbb{R}} = g(x, t), \quad x \in \partial \Omega, \ t \in \mathbb{R}.
\]

Let \( H^{2,1}_\gamma(\Omega \times \mathbb{R}) \) and \( H^{2,1}_\gamma(\partial \Omega \times \mathbb{R}) \) be the spaces of functions \( u(x, t) \) such that \( e^{-\gamma t}u \in H^{2,1}(\Omega \times \mathbb{R}) \) and \( e^{-\gamma t}u \in H^{2,1}(\partial \Omega \times \mathbb{R}) \) respectively. Denote the norms in these spaces by \( \| \cdot \|_{\gamma,2,1} \) and \( \langle \cdot \rangle_{\gamma,2,1} \) respectively. Here \( H^{2,1}(\Omega \times \mathbb{R}) \) and \( H^{2,1}(\partial \Omega \times \mathbb{R}) \) denote the anisotropic Sobolev spaces of functions that have two derivatives with respect to \( x \) and one derivative with respect to \( t \). The following theorem can be easily derived from the well-known results on solvability of general boundary value problems for parabolic equations (see, for instance, [AV]):

**Theorem 3.1.** Let \( K \) be a positive constant. There is a constant \( \gamma_0 > 0 \) depending only \( \Omega \) and \( K \) such that for arbitrary \( \gamma \geq \gamma_0 \) and \( k \in C(\overline{\Omega} \times [0, \infty)) \) with \( \|k\|_C \leq K \) the problem (3.1), (3.2) has a unique solution \( u \in H^{2,1}_\gamma(\Omega \times \mathbb{R}) \) for an arbitrary \( g \in H^{2,1}_\gamma(\partial \Omega \times \mathbb{R}) \). Moreover, the estimate

\[
\|u\|_{2,1}^2 + \|\partial_u u\|_{\gamma,0}^2 \leq C(g)^2_{\gamma,2,1}
\]

is valid for some constant \( C \) depending only on \( \Omega \) and \( K \).

Thereby, by analogy with 2.2, we can define the Dirichlet-to-Neumann map \( H : H^{2,1}_\gamma(\partial \Omega \times \mathbb{R}) \to H^{0}_\gamma(\partial \Omega \times \mathbb{R}) \). The inverse problem is to find \( k(x, t) \) from the given Dirichlet-to-Neumann map \( H \). As in the hyperbolic case we prove conditional stability:

**Theorem 3.2.** Suppose that \( k_j \in C^4(\overline{\Omega} \times [0, \infty)), \ j = 1, 2, \) are two functions such that \( \|k_j\|_{C^4} \leq K \), with some positive constant \( K \) (which determines the correctness class). There is \( \gamma_0 > 0 \) depending only on \( \Omega \) and \( K \) and such that, for every \( \gamma \geq \gamma_0 \), the corresponding Dirichlet-to-Neumann maps \( H_j \) are defined as operators from \( H^{2,1}_\gamma(\partial \Omega \times \mathbb{R}) \) to \( H^{0}_\gamma(\partial \Omega \times \mathbb{R}) \).
\[ \| k_1 - k_2 \|_{\gamma, 0} \leq \omega(\| H_1 - H_2 \|), \]

where
\[ \omega(\varepsilon) \sim C (\log \varepsilon^{-1})^{-1/(2n+3)}, \quad \varepsilon \to 0, \]

with \( C \) depending only on \( \Omega, K, \) and \( \gamma. \) Moreover, the result remains valid if we restrict the domain of \( H_j \) to the space \( \tilde{H}^{2,1}_\gamma(\partial \Omega \times \mathbb{R}) \) defined by analogy with the space \( \tilde{H}^{1}_\gamma(\partial \Omega \times \mathbb{R}) \) (see 2.1).

**Proof.** The proof repeats almost verbatim that of Theorem 2.2. The only difference is the choice of \( \zeta_j. \)

Applying the Fourier–Laplace transform to (3.1), we arrive at the equation
\[ i \theta \hat{u} - \Delta \hat{u} - \hat{k}(x, \theta) \hat{u}(x, \theta) = 0. \]

So now we have to choose \( \zeta_j \) from the conditions
\[ \zeta_1 + \zeta_2 = -i \xi, \quad \zeta_j \cdot i \theta = 0, \]
\[ |\zeta_j| \geq C_1 \| \hat{k}_j(\cdot, \theta) \|_{C^1(\Omega)}, \quad |\Im \zeta_j| \leq \sqrt{2}. \]

As in §2, searching \( \zeta_j \) in the form \( \zeta_j = -i \xi + (-1)^j (i \mu + \lambda), \) with \( \lambda, \mu \in \mathbb{R}^n, \lambda \cdot \xi = \mu \cdot \xi = 0, \) we obtain the system
\[ \lambda^2 - \mu^2 = \gamma + \frac{\xi^2}{4}, \quad 2 \lambda \cdot \mu = \sigma. \]

Take \( \lambda \) to be an arbitrary vector such that \( \lambda \cdot \xi = 0 \) and \( \lambda^2 = \gamma + \frac{\xi^2}{4} + |\sigma| + r^2, \quad r \geq \sqrt{2}. \) Afterwards, choose \( \mu \) from the conditions \( \mu \cdot \xi = 0, \mu^2 = \frac{|\sigma|}{2} + r^2, \) and \( \lambda \cdot \mu = \frac{\xi}{2}. \) We can fulfill the last condition by varying the angle between \( \lambda \) and \( \mu, \) since \( |\lambda||\mu| \geq |\sigma|. \)

In this case we have
\[ |\zeta_j|^2 = \frac{\xi^2}{2} + \gamma + |\sigma| + 2r^2, \]
so we can proceed as in the proof of Theorem 2.2.

§4. Solution of Memory Reconstruction Problems by the Newton Method

Here we consider the following two problems that appeared earlier (see [L]):

**Problem 1.** Find the pair of functions \((u, h)\) in the problem
\[
\begin{align*}
&u_t - Au - \int_0^t h(t - \tau) Bu(\tau) d\tau = f(t), \quad t \in [0, T]; \\
&u(0) = u_0
\end{align*}
\]
from the additional information
\[
\begin{align*}
&\Phi[u(t)] = g(t), \quad t \in [0, T].
\end{align*}
\]
Problem 2. Find the pair of functions \((u, h)\) in the problem

\[
\begin{align*}
    u_{tt} - Au - \int_0^t h(t - \tau) Bu(\tau) \, d\tau &= f(t), & t \in [0, T]; \\
    u(0) &= u_0, & u_t(0) = u_1
\end{align*}
\]

from the additional information

\[
\Phi[u(t)] = g(t), \quad t \in [0, T].
\]

Here \(A\) and \(B\) are closed densely-defined operators in a Banach space \(X\); \(u\) and \(f\) are functions from \([0, T]\) into \(X\); \(h\) and \(g\) are scalar functions on \([0, T]\); \(u_0\) and \(u_1\) are given elements on \(X\); and \(\Phi\) is a continuous linear functional on \(X\).

Let \(Y\) be the Banach space with the norm \(\|g\|_Y = \|g\|_X + \|Ag\|_X\). We consider Problems 1 and 2 in the following two cases:

(i) \(B(\text{dom } A) \subseteq \text{dom } A \quad \text{and} \quad B \in \mathcal{L}(Y)\);

(ii) \(\text{dom } B \supseteq \text{dom } A\) (for instance, \(B = A\)).

Suppose that \(A\) has an inverse operator and generates the strongly continuous semi-group \(e^{tA}\), \(t > 0\), (see \([F]\)). A typical example here is \(A = \Delta - \alpha\), \(\alpha > 0\), and \(X = L_2\). In the case (i) the following theorem is valid:

**Theorem 4.1.** If the data of the problem (4.1), (4.2) satisfy the smoothness conditions

\[
    f \in C^3([0, T], X), \quad g \in C^2([0, T]), \quad u_0 \in \text{dom } A^3, \quad f(0) \in \text{dom } A^2, \quad f'(t) \in \text{dom } A, \quad t \in [0, T];
\]

the agreement conditions

\[
    g(0) = \Phi[u_0], \quad g'(0) = \Phi[Au_0 + f(0)];
\]

and the condition \(\Phi[Bu_0] \neq 0\), then Problem 1 has a unique (global) solution \(u \in C^2([0, T], Y)\), \(h \in C([0, T])\). Moreover, the solution can be found by the Newton method and the convergence rate is described by the estimates

\[
\begin{align*}
    \|u_n - u\|_{C^2([0, T], Y)} &\leq C_1 \sigma T \frac{(C_2)^{2^n-1}}{2^n}, \\
    \|h_n - h\|_{C([0, T])} &\leq C_1 \sigma T \frac{(C_2)^{2^n-1}}{2^n},
\end{align*}
\]

where \(C_3 = (\|A^{-1}\| + 1)(T^3 - 1)(T - 1)^{-1}\), \(C_1 > 0\) and \(C_2 \in [0, 1]\) are arbitrary constants, and \(\sigma\) depends on \(C_1\) and \(C_2\).

Now, suppose that \(A\) has an inverse and generates the strongly continuous family \(\text{ch}(tA)\) (see \([IMF]\)). Then, for example, in the case (i) the following theorem is valid for Problem 2:

**Theorem 4.2.** If the data of the problem (4.3), (4.4) satisfy the smoothness conditions

\[
    f \in C^5([0, T], X), \quad g \in C^3([0, T]), \quad u_0 \in \text{dom } A^4, \quad u_1 \in \text{dom } A^3, \\
    f(0) \in \text{dom } A^2, \quad f'(0) \in \text{dom } A^2, \quad f''(0) \in \text{dom } A, \quad f'''(0) \in \text{dom } A;
\]
the agreement conditions

\[ g(0) = \Phi[u_0], \quad g'(0) = \Phi[u_1], \quad g''(0) = \Phi[Au_0 + f(0)], \]
\[ g'''(0) = \Phi[Au_1 + h(0)Bu_0 + f'(0)]; \]

and the condition \( \Phi[Bu_0] \neq 0 \), then Problem 2 has a unique (global) solution \( u \in C^4([0,T], Y) \), \( h \in C^1([0,T]) \). Moreover, the solution can be found by the Newton method and the convergence rate is described by the estimates

\[ \|u_n - u\|_{C^4([0,T], Y)} \leq C_1 C_3 e^{\sigma T} \frac{(C_2)^{2n-1}}{2^n}, \]
\[ \|h_n - h\|_{C([0,T])} \leq C_1 C_4 e^{\sigma T} \frac{(C_2)^{2n-1}}{2^n}, \]

where \( C_3 = (\|A^{-1}\| + 1) (T^5 - 1) (T - 1)^{-1}, \ C_4 = T + 1, \ C_1 > 0 \) and \( C_2 \in [0, 1] \) are arbitrary constants, and \( \sigma \) depends on \( C_1 \) and \( C_2 \).

We can prove similar theorems (with slightly different conditions on the data and the same estimates) for Problems 1 and 2 in the case (ii).

The proofs of these results are rather technical and bulky, so we refer the reader to [BK]. We only observe that we reduce Problems 1 and 2 to a nonlinear system of integral equations as in [B1, L] and then apply the Newton method.

As an example, consider the Cauchy problem

\[ u_{tt} - \Delta u - \int_0^t h(t - \tau) u(x, \tau) \, d\tau = 0, \quad x \in \mathbb{R}^3, \ 0 < t < T, \]
\[ u(x, 0) = 0, \ u_t(x, 0) = \delta(x), \quad x \in \mathbb{R}^3, \]

with \( h \in C[0, T] \). A solution to (4.5) is understood in the distributional sense and has the form \( u(x, t) = E(x, t) + v(x, t) \), where \( E(x, t) = \frac{\delta(t)}{2\pi} \delta(t^2 - |x|^2) \) and the scattered wave \( v(x, t) \) is continuous.

The inverse problem is to find the memory \( h(t) \) from the measurements of the scattered wave at the origin, i.e., from the information

\[ f(t) = v(0, t), \quad 0 < t < T. \]

Adapting the scheme of the proof of Theorem 4.2 and using the Newton method, we can prove global unique solvability in the inverse problem (4.5), (4.6):

**Theorem 4.3.** If \( f \in C^1[0,T] \) and \( f(0) = 0 \) then the inverse problem (4.5), (4.6) has a unique solution \( h \in C[0,T] \).

A more complicated problem

\[ u_{tt} - \Delta u - \int_0^t h(t - \tau) \Delta u(x, \tau) \, d\tau = 0, \quad x \in \mathbb{R}^3, \ 0 < t < T, \]
\[ u(x, 0) = 0, \ u_t(x, 0) = \delta(x), \quad x \in \mathbb{R}^3, \]
\[ f(t) = v(x_0, t), \quad |x_0| < t < T, \]

where \( x_0 \) in an arbitrary fixed point in \( \mathbb{R}^3 \), is considered in [BKK].

Observe that here we can proceed immediately without semigroup theory.
REFERENCES


Inverse Problems for Equations with Memory


